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Cambridge International AS & A Level Mathematics:

Pure Mathematics 1

Worked Solutions Manual

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Contents

The items in orange are available on the Elevate edition that accompanies this book.

How to use this book

This book contains worked solutions to the questions in the *Cambridge International AS & A Level Mathematics: Pure Mathematics 1 Coursebook*. Both the book and accompanying Elevate edition include the solutions to the chapter exercises. You will find the solutions to the end-of-chapter review exercises, cross-topic review exercises and practice exam-style paper on the Elevate edition only.

Most of the chapter exercises include questions to help develop your fluency in solving a particular type of problem by practising the procedure several times. Rather than providing worked solutions for all of these questions, we have included a worked solution for one or two of the fluency questions, which can then be used for guidance about the steps required for the related questions. The aim of this is to encourage you to develop as a confident, independent thinker.

Each solution shows you step-by-step how to solve the question. You will be aware that often questions can be solved by multiple different methods. In this book, we provide a single method for each solution. Do not be disheartened if the working in a solution does not match your own working; you may not be wrong but simply using a different method. It is good practice to challenge yourself to think about the methods you are using and whether there may be alternative methods.

All worked solutions in this resource have been written by the author. In examinations, the way marks are awarded may be different.

Additional guidance is included in **Commentary** boxes throughout the book. These boxes often clarify common misconceptions or areas of difficulty.

E Some questions in the coursebook go beyond the syllabus. We have indicated these solutions with a red line to the left of the text.

Chapter 1
Quadratics

EXERCISE 1A

1 a $x^2 + 3x - 10 = 0$

$(x+5)(x-2) = 0$

$x+5 = 0$ or $x-2 = 0$

$x = -5$ or $x = 2$

f $x(10x-13) = 3$

$10x^2 - 13x = 3$

$10x^2 - 13x - 3 = 0$

$(5x+1)(2x-3) = 0$

$5x+1 = 0$ or $2x-3 = 0$

$x = -\frac{1}{5}$ or $x = \frac{3}{2}$

2 c $\frac{5x+1}{4} - \frac{2x-1}{2} = x^2$

Multiply both sides by 4

$5x + 1 - 2(2x-1) = 4x^2$

Multiplying by 8 will give the same answer.

$4x^2 - x - 3 = 0$

$(4x+3)(x-1) = 0$

$4x+3 = 0$ or $x-1 = 0$

$x = -\frac{3}{4}$ or $x = 1$

f $\frac{3}{x+2} + \frac{1}{x-1} = \frac{1}{(x+1)(x+2)}$

Multiply both sides by $(x+1)(x+2)(x-1)$

$3(x+1)(x-1) + (x+2)(x+1) = 1(x-1)$

$3(x^2-1) + x^2 + 3x + 2 = x - 1$

$3x^2 - 3 + x^2 + 3x + 2 = x - 1$

$4x^2 + 2x = 0$

Do **NOT** be tempted to divide both sides by x next.
This will lose the solution $x = 0$.

Factorise

$2x(2x+1) = 0$

$2x = 0$ or $2x+1 = 0$

$x = 0$ or $x = -\frac{1}{2}$

3 a $\frac{3x^2 + x - 10}{x^2 - 7x + 6} = 0$

Multiply both sides by $x^2 - 7x + 6$

$3x^2 + x - 10 = 0$

$(3x-5)(x+2) = 0$

$3x-5 = 0$ or $x+2 = 0$

$x = \frac{5}{3}$ or $x = -2$

Always substitute your answers back into the original equations to make sure that no denominators evaluate to 0.

d $\frac{x^2 - 2x - 8}{x^2 + 7x + 10} = 0$

Multiply both sides by $x^2 + 7x + 10$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$(x-4) = 0$ or $(x+2) = 0$

$x = 4$ or $x = -2$

If $x = -2$, the denominator becomes

$(-2)^2 + 7(-2) + 10$

Which evaluates to zero so $x = -2$ is **NOT** a solution

The only solution is $x = 4$.

f $\frac{2x^2 + 9x - 5}{x^4 + 1} = 0$

Multiply both sides by $x^4 + 1$

$2x^2 + 9x - 5 = 0$

$(2x-1)(x+5) = 0$

$2x-1 = 0$ or $x+5 = 0$

$x = \frac{1}{2}$ or $x = -5$

Check: neither of these solutions, when substituted back into the fraction evaluate to zero so both are valid.

4 c $2^{(x^2-4x+6)} = 8$

Rewrite 8 as 2^3

$2^{(x^2-4x+6)} = 2^3$

Equating powers of 2 gives:

$x^2 - 4x + 6 = 3$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x - 1 = 0$ or $x - 3 = 0$

$x = 1$ or $x = 3$

f $(x^2 - 7x + 11)^8 = 1$

Find the eighth root of both sides of the equation.

$[(x^2 - 7x + 11)^8]^{\frac{1}{8}} = [1]^{\frac{1}{8}}$

$x^2 - 7x + 11 = \pm 1$

Don't forget the two roots here.

$x^2 - 7x + 10 = 0$ or $x^2 - 7x + 12 = 0$

$(x-2)(x-5) = 0$ or $(x-3)(x-4) = 0$

$x = 2$ or $x = 3$ or $x = 4$ or $x = 5$

5 a Using Pythagoras:

$(2x)^2 + (2x+1)^2 = 29^2$

$4x^2 + 4x^2 + 4x + 1 = 841$

$8x^2 + 4x - 840 = 0$

Divide both sides by the common factor of 4:

$2x^2 + x - 210 = 0$ Shown

b $(x-10)(2x+21) = 0$

$x - 10 = 0$ or $2x + 21 = 0$

$x = 10$ or $x = -10.5$

The sides of the triangle are 20 cm, 21 cm and 29 cm.

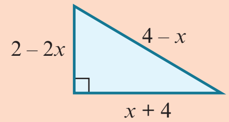
Check that your answers satisfy the original equation.

$(2(10))^2 + (2(10)+1)^2 = 29^2$

$400 + 441 = 841$

$841 = 841$

Do not automatically reject negative values for x. In this example $x = -1$ but this gives positive lengths when substituted into the sides of the triangle.



6 Area of a trapezium is $\frac{1}{2}(a+b)h$

$\frac{1}{2}[(x-1) + (x+3)]x = 35.75$

Multiply both sides by 4:

$2[(x-1) + (x+3)]x = 143$

$2[2x+2]x = 143$

$4x^2 + 4x = 143$

$4x^2 + 4x - 143 = 0$

$(2x-11)(2x+13) = 0$

$2x - 11 = 0$ or $2x + 13 = 0$

$x = 5.5$ or $x = -6.5$

Since x is the length of one of the sides of the trapezium, x must be positive.

$x = 5.5$

7 $(x^2 - 11x + 29)^{(6x^2+x-2)} = 1$

Case 1: for any number a we have $a^0 = 1$, so solve $6x^2 + x - 2 = 0$, for some solutions.

$6x^2 + x - 2 = 0$

$(2x-1)(3x+2) = 0$

$x = \frac{1}{2}$ or $x = -\frac{2}{3}$

Case 2: for any number b we have $1^b = 1$, so solve $x^2 - 11x + 29 = 1$ for more solutions.

$x^2 - 11x + 29 = 1$

$x^2 - 11x + 28 = 0$

$(x-4)(x-7) = 0$

$x - 4 = 0$ or $x - 7 = 0$

$x = 4$ or $x = 7$

Case 3: $(-1)^{2b} = 1$ **for any number** b**, so solve** $x^2 - 11x + 29 = -1$ **to see whether the numbers we get lead to** $6x^2 + x - 2$ **being an even number.**

$x^2 - 11x + 29 = -1$

$x^2 - 11x + 30 = 0$

$(x - 6)(x - 5) = 0$

$x - 6 = 0$ or $x - 5 = 0$

$x = 6$ or $x = 5$

Substituting $x = 6$ into $6x^2 + x - 2$

$6(6)^2 + 6 - 2 = 220$

This gives an even number, so $x = 6$ is a solution

Substituting $x = 5$ into $6x^2 + x - 2$

$6(5)^2 + 5 - 2 = 153$

This gives an odd number, so $x = 5$ is **not** a solution

Real number solutions are: $x = -\frac{2}{3}$, $\frac{1}{2}$, 4, 6 and 7.

EXERCISE 1B

1 **a** $x^2 - 6x = (x - 3)^2 - 3^2$

$= (x - 3)^2 - 9$

g $x^2 + 7x + 1 = \left(x + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 1$

$= \left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$

2 **b** $3x^2 - 12x - 1$

Take out a factor of 3 from the first two terms:

$3(x^2 - 4x) - 1$

Complete the square:

$3[(x - 2)^2 - 4] - 1$

$3(x - 2)^2 - 13$

3 **c** $4 - 3x - x^2$

$4 - (3x + x^2)$

$4 - \left[\left(\frac{3}{2} + x\right)^2 - \left(\frac{3}{2}\right)^2\right]$

$4 + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2} + x\right)^2$

$\frac{25}{4} - \left(x + \frac{3}{2}\right)^2$

4 **b** $3 - 12x - 2x^2$

$3 - 2(6x + x^2)$

$3 - 2[(3 + x)^2 - 3^2]$

$3 - 2(3 + x)^2 + 18$

$21 - 2(x + 3)^2$

5 **a** $9x^2 - 6x - 3$

Using an algebraic method:

$9x^2 - 6x - 3 = (ax + b)^2 + c$

$= a^2x^2 + 2abx + b^2 + c$

$9 = a^2$...., $-6 = 2ab$ ...., $-3 = b^2 + c$ ....

So $a = \pm 3$

If $a = 3$, $-6 = 6b$ so $b = -1$, then:

$-3 = (-1)^2 + c$ so $c = -4$

If $a = -3$, $-6 = -6b$ so $b = 1$

$-3 = 1^2 + c$ so $c = -4$

$9x^2 - 6x - 3 = (3x - 1)^2 - 4 = (-3x + 1)^2 - 4$

6 **a** $x^2 + 8x - 9 = 0$

$(x + 4)^2 - 16 - 9 = 0$

$(x + 4)^2 = 25$

Square root both sides:

$x + 4 = \pm 5$

$x = -9$, or $x = 1$

7 **a** $x^2 + 4x - 7 = 0$

$(x + 2)^2 - 4 - 7 = 0$

$(x + 2)^2 = 11$

$x + 2 = \pm\sqrt{11}$

$x = -2 \pm \sqrt{11}$

e $2x^2 + 6x + 3 = 0$

$$2\left[\left(x+\frac{3}{2}\right)^2 - \frac{9}{4}\right] + 3 = 0$$

$$2\left(x+\frac{3}{2}\right)^2 - \frac{9}{2} + 3 = 0$$

$$2\left(x+\frac{3}{2}\right)^2 = \frac{3}{2}$$

$$\left(x+\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$x+\frac{3}{2} = \pm\frac{\sqrt{3}}{2}$$

$$x = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} \text{ or } x = \frac{-3\pm\sqrt{3}}{2}$$

8 $\dfrac{5}{x+2} + \dfrac{3}{x-4} = 2$

Multiply all terms by $(x+2)(x-4)$:

$5(x-4) + 3(x+2) = 2(x+2)(x-4)$

$5x - 20 + 3x + 6 = 2x^2 - 4x - 16$

$2x^2 - 12x - 2 = 0$ dividing both sides by 2 gives:

$$x^2 - 6x - 1 = 0$$

$$(x-3)^2 - 3^2 - 1 = 0$$

$$(x-3)^2 = 10$$

$$x - 3 = \pm\sqrt{10}$$

$$x = 3 \pm \sqrt{10}$$

9 Using Pythagoras:

$$(2x+5)^2 + x^2 = 10^2$$

$$5x^2 + 20x - 75 = 0$$

$$x^2 + 4x - 15 = 0$$

$$(x+2)^2 - 2^2 - 15 = 0$$

$$(x+2)^2 = 19$$

$$x + 2 = \pm\sqrt{19}$$

$x = \sqrt{19} - 2$ or

$x = -\sqrt{19} - 2$ (reject as a negative value is not valid for the sides of a triangle.)

$x = \sqrt{19} - 2$

10 $(3x^2 + 5x - 7)^4 = 1$

Taking the 4th root of both sides gives:

$3x^2 + 5x - 7 = \pm 1$

Either: $3x^2 + 5x - 7 = 1$

$$3x^2 + 5x - 8 = 0$$

$$3\left[\left(x+\frac{5}{6}\right)^2 - \frac{25}{36}\right] - 8 = 0$$

$$3\left(x+\frac{5}{6}\right)^2 - \frac{25}{12} - 8 = 0$$

$$3\left(x+\frac{5}{6}\right)^2 = \frac{121}{12}$$

$$\left(x+\frac{5}{6}\right)^2 = \frac{121}{36}$$

$$x+\frac{5}{6} = \pm\sqrt{\left(\frac{121}{36}\right)}$$

$$x+\frac{5}{6} = \frac{11}{6} \text{ or } x+\frac{5}{6} = -\frac{11}{6}$$

$$x = 1 \text{ or } -\frac{8}{3}$$

Or: $3x^2 + 5x - 7 = -1$

$$3x^2 + 5x - 6 = 0$$

$$3\left[\left(x+\frac{5}{6}\right)^2 - \frac{25}{36}\right] - 6 = 0$$

$$3\left(x+\frac{5}{6}\right)^2 - \frac{25}{12} - 6 = 0$$

$$3\left(x+\frac{5}{6}\right)^2 = \frac{97}{12}$$

$$\left(x+\frac{5}{6}\right)^2 = \frac{97}{36}$$

$$x+\frac{5}{6} = \pm\frac{\sqrt{97}}{6}$$

$$x = \frac{1}{6}\left(-5-\sqrt{97}\right) \text{ or } \frac{1}{6}\left(\sqrt{97}-5\right)$$

$$x = -\frac{8}{3},\ 1,\ \frac{1}{6}\left(-5-\sqrt{97}\right),\ \frac{1}{6}\left(\sqrt{97}-5\right)$$

11 $y = \left(\sqrt{3}\right)x - \dfrac{49x^2}{9000}$

a The range is the maximum value of x. This is when $y = 0$.

$$\left(\sqrt{3}\right)x - \frac{49x^2}{9000} = 0 \text{ ....[1]}$$

$$9000\left(\sqrt{3}\right)x - 49x^2 = 0$$

$$49\left[\left(\frac{9000\sqrt{3}}{98} - x\right)^2 - \left(\frac{9000\sqrt{3}}{98}\right)^2\right] = 0$$

$$\left(\frac{9000\sqrt{3}}{98} - x\right)^2 - \left(\frac{9000\sqrt{3}}{98}\right)^2 = 0$$

$$\left(\frac{9000\sqrt{3}}{98} - x\right)^2 = \left(\frac{9000\sqrt{3}}{98}\right)^2$$

Square root both sides

$$\frac{9000\sqrt{3}}{98} - x = \pm\frac{9000\sqrt{3}}{98}$$

$$x = \frac{9000\sqrt{3}}{49} \text{ or } x = 0 \text{ reject}$$

$$x = \frac{9000\sqrt{3}}{49} \approx 318 \text{ m (3 significant figures)}$$

Factorising is another possible method to solve Equation [1]

$x\left(9000\sqrt{3} - 49x\right) = 0$

Either $x = 0$, (*reject*) or $9000\sqrt{3} - 49x = 0$

$x = \frac{9000\sqrt{3}}{49} \approx 318$ m (3 significant figures)

b The maximum height reached is the largest value of y.

This occurs when $x = \frac{9000\sqrt{3}}{98}$ since the highest point on the graph is mid-way in the flight. So,

$\frac{9000\sqrt{3}}{49}$ divided by 2 is $\frac{9000\sqrt{3}}{98}$

Substituting into $y = \left(\sqrt{3}\right)x - \frac{49x^2}{9000}$ gives:

$$y = \left(\sqrt{3}\right)\frac{9000\sqrt{3}}{98} - \frac{49}{9000}\left(\frac{9000\sqrt{3}}{98}\right)^2$$

$$y = \frac{27000}{98} - \frac{13500}{98}$$

$y = 138$ m to 3 significant figures.

There is another way to approach Question 11, which you will meet in Chapter 8.

EXERCISE 1C

1 a $x^2 - 10x - 3 = 0.$

Using $a = 1$, $b = -10$ and $c = -3$ in the quadratic formula gives:

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$

$$x = \frac{10 + \sqrt{112}}{2} \text{ or } x = \frac{10 - \sqrt{112}}{2}$$

$x = 10.29$ or $x = -0.29$ (to 3 sf)

2 $x(3x - 2) = 63$

$3x^2 - 2x - 63 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-63)}}{2 \times 3}$$

$$x = \frac{2 + \sqrt{760}}{6} \text{ or } x = \frac{2 - \sqrt{760}}{6}$$

$x = 4.928$ or $x = -4.261$ (reject)

$x = 4.93$ to 3 significant figures.

3 $x(2x - 4) = (x + 1)(5 - x)$

$3x^2 - 8x - 5 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 3 \times (-5)}}{2 \times 3}$$

$$x = \frac{8 + \sqrt{124}}{6} \text{ or } x = \frac{8 - \sqrt{124}}{6}$$

$x = 3.189$ or $x = -0.5226$ (reject)

$x = 3.19$ to 3 significant figures.

4 $\frac{5}{x - 3} + \frac{2}{x + 1} = 1$

Multiplying both sides by $(x - 3)(x + 1)$ gives:

$5(x + 1) + 2(x - 3) = 1(x - 3)(x + 1)$

$x^2 - 9x - 2 = 0$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$

$$x = \frac{9 + \sqrt{89}}{2} \text{ or } x = \frac{9 - \sqrt{89}}{2}$$

$x = 9.22$ or $x = -0.217$ to 3 significant figures.

5 $ax^2 - bx + c = 0$

$x = \dfrac{-(-b) \pm \sqrt{(-b)^2 - 4 \times a \times c}}{2 \times a}$

$x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$ or $\dfrac{b}{2a} \pm \dfrac{\sqrt{(b^2 - 4ac)}}{2a}$

Compare with $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ or

$\dfrac{-b}{2a} \pm \dfrac{\sqrt{(b^2 - 4ac)}}{2a}$

The solutions both increase by $\dfrac{b}{a}$

EXERCISE 1D

1 b $x + 4y = 6$ ....[1]

$x^2 + 2xy = 8$ ....[2]

> It is best to avoid fractions (if possible) when using substitution.

$x = 6 - 4y$

Substitute into [2] gives:

$(6 - 4y)^2 + 2(6 - 4y)y = 8$

$8y^2 - 36y + 28 = 0$

Divide by 4

$2y^2 - 9y + 7 = 0$

$(y - 1)(2y - 7) = 0$

$y = 1$ or $y = \dfrac{7}{2}$

Substitute into [1]

If $y = 1$ then $x = 2$

If $y = \dfrac{7}{2}$ then $x = -8$

> Always substitute back into the linear equation.

Solutions are $\left(-8, \dfrac{7}{2}\right)$ and $(2, 1)$

f $4x - 3y = 5$ ....[1]

$x^2 + 3xy = 10$ ....[2]

> Before you start, look for the least complicated method.

Method 1

Make x the subject of [1]

$x = \dfrac{5 + 3y}{4}$

Substitute into [2]

$\left(\dfrac{5 + 3y}{4}\right)^2 + 3\left(\dfrac{5 + 3y}{4}\right)y = 10$

$\dfrac{(5 + 3y)^2}{16} + \dfrac{3y(5 + 3y)}{4} = 10$

$(5 + 3y)^2 + 12y(5 + 3y) = 160$

$45y^2 + 90y - 135 = 0$

$y^2 + 2y - 3 = 0$

$(y + 3)(y - 1) = 0$

$y = -3$ or 1

Substitute back into [1]

$4x - 3(-3) = 5$ and $4x - 3(1) = 5$

$x = -1$ $\quad$ $x = 2$

Solutions are $(-1, -3)$, $(2, 1)$

The alternative method below is much easier:

Method 2

From [1], multiply $4x - 3y = 5$ by x and then add the new equation to [2]

$4x^2 - 3xy = 5x$

$x^2 + 3xy = 10$

Adding gives $5x^2 = 5x + 10$ or $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or -1

Substituting back into the linear equation [1] gives

$4(2) - 3y = 5$ and $4(-1) - 3y = 5$

$y = 1$ $\quad$ $y = -3$

Solutions are (–1, –3), (2, 1)

n $x + 2y = 5$ ....[1]

$x^2 + y^2 = 10$ ....[2]

A common mistake is to rewrite [2] as $x + y = \sqrt{10}$.

From [1] $\quad x = 5 - 2y$

Substitute for x in [2]

$$(5 - 2y)^2 + y^2 = 10$$
$$5y^2 - 20y + 15 = 0$$
$$y^2 - 4y + 3 = 0$$
$$(y - 3)(y - 1) = 0$$
$$y = 3 \text{ or } 1$$

Substituting back into [1] gives:

$x + 2(3) = 5$ and $x + 2(1) = 5$

$x = -1$ and $x = 3$

Solutions are (–1, 3), (3, 1)

2 **a** Let the numbers be x and y

$x + y = 26$ ....[1]

$xy = 153$ ....[2]

From [1] $x = 26 - y$

Substitute for x into [2]

$$(26 - y)y = 153$$
$$y^2 - 26y + 153 = 0$$
$$(y - 9)(y - 17) = 0$$
$$y = 9 \text{ or } 17$$

Substituting into [1] gives:

$x = 17$ or 9

The two numbers are 9 and 17

b [1] remains the same and [2] becomes

$xy = 150$ ....[2]

[2] now becomes:

$(26 - y)y = 150$ which simplifies to:

$y^2 - 26y + 150 = 0$

Solving using the formula gives:

$$y = \frac{-(-26) \pm \sqrt{(-26)^2 - 4 \times 1 \times (150)}}{2 \times 1}$$

$y = 13 - \sqrt{19}$ and $y = 13 + \sqrt{19}$

Leading to the two numbers $13 - \sqrt{19}$ and $13 + \sqrt{19}$

3 Let the lengths of the sides of the rectangle be x and y.

$2x + 2y = 15.8$ ....[1]

$xy = 13.5$ ....[2]

From [1] $x = 7.9 - y$

Substitute for x in [2]

$$(7.9 - y)y = 13.5$$
$$y^2 - 7.9y + 13.5 = 0$$
$$y = \frac{-(-7.9) \pm \sqrt{(-7.9)^2 - 4 \times 1 \times (13.5)}}{2 \times 1}$$
$$y = \frac{27}{5} \text{ or } \frac{5}{2}$$

Substituting $y = \frac{27}{5}$ into [2] gives $x = \frac{5}{2}$

Substituting $y = \frac{5}{2}$ into [2] gives $x = \frac{27}{5}$

The lengths of the sides of the rectangle are $2\frac{1}{2}$ cm and $5\frac{2}{5}$ cm.

4 Let the sides of the squares be x cm and y cm.

Total perimeter is $4x + 4y = 50$ ....[1]

Total area is $x^2 + y^2 = 93.25$ ....[2]

From [1] $\quad x = \frac{25 - 2y}{2}$

Substitute for x in [2]

$$\left(\frac{25 - 2y}{2}\right)^2 + y^2 = 93.25$$
$$(25 - 2y)^2 + 4y^2 = 373$$
$$8y^2 - 100y + 252 = 0$$
$$2y^2 - 25y + 63 = 0$$
$$y = \frac{-(-25) \pm \sqrt{(-25)^2 - 4 \times 2 \times (63)}}{2 \times 2}$$
$$y = 9 \text{ or } 3\frac{1}{2}$$

Substitute $y = 9$ into [1] gives $x = 3\frac{1}{2}$

Substituting $y = 3\frac{1}{2}$ into [1] gives $x = 9$

The squares are each of side length $3\frac{1}{2}$ cm and 9 cm

5 Let the two radii be x and y

$2\pi x + 2\pi y = 36\pi$ ....[1]

$\pi x^2 + \pi y^2 = 170\pi$ ....[2]

Simplifying each equation:

$x + y = 18$ ....[1]

$x^2 + y^2 = 170$ ....[2]

From [1] $x = 18 - y$

Substituting for x in [2]

$(18 - y)^2 + y^2 = 170$

$y^2 - 18y + 77 = 0$

$(y - 11)(y - 7) = 0$

$y = 11$ or 7

Substitute $y = 11$ into [1] gives $x = 7$

Substitute $y = 7$ into [1] gives $x = 11$

The radii are 7 cm and 11 cm.

6 $x + y = 20.5$ ....[1]

$5xy = 360$ ....[2]

From [1] $x = 20.5 - y$

Substitute for x into [2]

$5(20.5 - y)y = 360$

$5y^2 - 102.5y + 360 = 0$

$$y = \frac{-(-102.5) \pm \sqrt{(-102.5)^2 - 4 \times 5 \times (360)}}{2 \times 5}$$

$y = 16$ or $\frac{9}{2}$

Substituting $y = 16$ into [1] gives $x = 4\frac{1}{2}$

Substituting $y = 4\frac{1}{2}$ into [1] gives $x = 16$

$x = 4\frac{1}{2}$, $y = 16$ or $x = 16$, $y = 4\frac{1}{2}$

7 $h + r = 18$ ....[1]

$\frac{1}{2}(4\pi r^2) + \pi r^2 + 2\pi rh = 205\pi$ ....[2] which simplifies to:

$3r^2 + 2rh - 205 = 0$

From [1] $h = 18 - r$

Substitute for h in [2]

$3r^2 + 2r(18 - r) - 205 = 0$

$r^2 + 36r - 205 = 0$

$(r - 5)(r + 41) = 0$

$r = 5$ or $r = -41$ (reject)

Substituting $r = 5$ into [1] gives $h = 13$

Solution $r = 5$, $h = 13$

8 **a** $y = 2 - x$ ....[1]

$5x^2 - y^2 = 20$ ....[2]

Substitute for y in [2]

$5x^2 - (2 - x)^2 = 20$

$x^2 + x - 6 = 0$

$(x - 2)(x + 3) = 0$

$x = 2$ or $x = -3$

Substituting $x = 2$ into [1] gives $y = 0$

Substituting $x = -3$ into [1] gives $y = 5$

A is at $(2, 0)$ and B is at $(-3, 5)$ (or vice versa)

b Using Pythagoras $AB = \sqrt{(2 - -3)^2 + (0 - 5)^2}$

$AB = \sqrt{50}$

The length of AB is $5\sqrt{2}$

9 **a** $2x + 5y = 1$ ....[1]

$x^2 + 5xy - 4y^2 + 10 = 0$ ....[2]

From [1] $x = \frac{1 - 5y}{2}$

Substitute for x in [2]

$$\left(\frac{1 - 5y}{2}\right)^2 + 5\left(\frac{1 - 5y}{2}\right)y - 4y^2 + 10 = 0$$

$(1 - 5y)^2 + 10(1 - 5y)y - 16y^2 + 40 = 0$

$-41y^2 + 41 = 0$

$-41(y^2 - 1) = 0$

$-41(y - 1)(y + 1) = 0$

$y = 1$ or $y = -1$

Substituting $y = 1$ into [1] gives $x = -2$

Substituting $y = -1$ into [1] gives $x = 3$

A is at $(-2, 1)$ and B is at $(3, -1)$ or vice-versa.

b Midpoint of AB is at $\left[\left(\frac{-2+3}{2}\right), \left(\frac{1-1}{2}\right)\right]$ or $\left(\frac{1}{2}, 0\right)$

11 $7y - x = 25$ ....[1]

$x^2 + y^2 = 25$ ....[2]

From [1] $x = 7y - 25$

Substitute for x in [2]

$(7y-25)^2 + y^2 = 25$

$y^2 - 7y + 12 = 0$

$(y-3)(y-4) = 0$

$y = 3$ or $y = 4$

Substituting $y = 3$ into [1] gives $x = -4$

Substituting $y = 4$ into [1] gives $x = 3$

A is at $(-4, 3)$ and B is at $(3, 4)$ or vice-versa.

Midpoint of AB is at $\left[\left(\frac{-4+3}{2}\right), \left(\frac{3+4}{2}\right)\right]$ or $\left(-\frac{1}{2}, \frac{7}{2}\right)$

Gradient of line $AB = \frac{3-4}{-4-3}$ or $\frac{1}{7}$

Gradient of a line perpendicular to AB is -7

Equation of perpendicular bisector of AB is the line with gradient -7 which passes through the point $\left(-\frac{1}{2}, \frac{7}{2}\right)$

Using $(y - y_1) = m(x - x_1)$

$\left(y - \frac{7}{2}\right) = -7\left(x - -\frac{1}{2}\right)$

$2y - 7 = -14\left(x + \frac{1}{2}\right)$

$2y - 7 = -14x - 7$

The equation is $7x + y = 0$

12 $y = x + 1$ ....[1]

$x^2 - y = 5$ ....[2]

From [1], substitute for y in [2]

$x^2 - (x+1) = 5$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

Substituting $x = 3$ into [1] gives $y = 4$

Substituting $x = -2$ into [1] gives $y = -1$

A is at $(-2, -1)$ and B is at $(3, 4)$

As $AP : PB = 4 : 1$

Point P is $\frac{4}{5}$ of the way along AB

P is at $\left\{\left[-2 + \frac{4}{5}(3 - -2)\right], \left[-1 + \frac{4}{5}(4 - -1)\right]\right\}$

P is at $(2, 3)$

14 a Let the parts be x and y.

$x + y = 10$ ....[1]

$x^2 - y^2 = 60$ ....[2]

From [1] $x = 10 - y$

Substitute for x in [2]

$(10-y)^2 - y^2 = 60$

$-20y = -40$

$y = 2$

Therefore $x = 8$

b $x + y = N$ ....[1]

$x^2 - y^2 = D$ ....[2]

$(N-y)^2 - y^2 = D$

$N^2 - 2Ny = D$

$2Ny = N^2 - D$

$y = \frac{N^2}{2N} - \frac{D}{2N}$

$y = \frac{N}{2} - \frac{D}{2N}$

$x = N - \left(\frac{N}{2} - \frac{D}{2N}\right)$

$x = \frac{N}{2} + \frac{D}{2N}$

The two parts are $\frac{N}{2} + \frac{D}{2N}$ and $\frac{N}{2} - \frac{D}{2N}$

EXERCISE 1E

1 a Method 1 (Substitution)

$x^4 - 13x^2 + 36 = 0$

Let $y = x^2$ then:

$y^2 - 13y + 36 = 0$

$(y-4)(y-9) = 0$

$y = 4$ or $y = 9$

$x^2 = 4$ or $x^2 = 9$

$x = \pm 2$ or $x = \pm 3$

Method 2 (Factorise directly)

$(x^2 - 4)(x^2 - 9) = 0$

$x^2 = 4$ or $x^2 = 9$

$x = \pm 2$ or $x = \pm 3$

l $\frac{8}{x^6} + \frac{7}{x^3} = 1$

$8 + 7x^3 = x^6$

$x^6 - 7x^3 - 8 = 0$

$(x^3 - 8)(x^3 + 1) = 0$

$x^3 = 8$ or $x^3 = -1$

$x = 2$ or $x = -1$

2 b $\sqrt{x}\left(\sqrt{x} + 1\right) = 6$

$x + \sqrt{x} - 6 = 0$

Let $y = \sqrt{x}$ then:

$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$

$y = -3$ or $y = 2$

$\sqrt{x} = -3$ (no solutions as $\sqrt{x}$ is never negative)

$\sqrt{x} = 2$

$x = 4$

f $3\sqrt{x} + \frac{5}{\sqrt{x}} = 16$ multiply both sides by $\sqrt{x}$

$3x - 16\sqrt{x} + 5 = 0$

Let $y = \sqrt{x}$ then:

$3y^2 - 16y + 5 = 0$

$(3y-1)(y-5) = 0$

$y = \frac{1}{3}$ or $y = 5$

$\sqrt{x} = \frac{1}{3}$ or $\sqrt{x} = 5$

$x = \frac{1}{9}$ or $x = 25$

3 a $y = 2\sqrt{x}$ ……………….[1]

$3y = x + 8$ ……………[2]

From [1], substitute for y in [2]

$3\left(2\sqrt{x}\right) = x + 8$

$x - 6\sqrt{x} + 8 = 0$

b Let $y = \sqrt{x}$ then:

$y^2 - 6y + 8 = 0$

$(y-2)(y-4) = 0$

$y = 2$ or $y = 4$

$\sqrt{x} = 2$ or $\sqrt{x} = 4$

$x = 4$ or $x = 16$

Substituting $x = 4$ into [2] gives $y = 4$

Substituting $x = 16$ into [2] gives $y = 8$

A is at $(4, 4)$ and B is at $(16, 8)$ or vice-versa.

c Using Pythagoras $AB = \sqrt{(16-4)^2 + (8-4)^2}$

$AB = 4\sqrt{10}$

4 $y = ax + b\sqrt{x} + c$

Substituting $x = 0$ and $y = 7$ into $y = ax + b\sqrt{x} + c$ gives:

$c = 7$, so

$y = ax + b\sqrt{x} + 7$

Substituting $x = 1$ and $y = 0$ into $y = ax + b\sqrt{x} + 7$

$a + b = -7$ ……………….. [1]

Substituting $x = \frac{49}{4}$ and $y = 0$ into $y = ax + b\sqrt{x} + 7$

$0 = \frac{49}{4}a + \frac{7}{2}b + 7$ …………[2]

Simplifying [2]

$7a + 2b = -4$ ……….........[3]

Multiply [1] by 2 and subtract [3]

$-5a = -10$

$a = 2$

Substituting $a = 2$ into [1]

$b = -9$

So, $a = 2$, $b = -9$, $c = 7$

5 $y = a(2^{2x}) + b(2^x) + c$

Substituting $x = 0$ and $y = 90$ into

$y = a(2^{2x}) + b(2^x) + c$

$90 = a(2^0) + b(2^0) + c$

$90 = a + b + c$ .... [1]

Substituting $x = 2$ and $y = 0$ into

$y = a(2^{2x}) + b(2^x) + c$

$0 = a(2^4) + b(2^2) + c$

$0 = 16a + 4b + c$ .... [2]

Subtract [1] from [2] and simplify

$-30 = 5a + b$ .... [3]

Substituting $x = 4$ and $y = 0$ into

$y = a(2^{2x}) + b(2^x) + c$

$0 = a(2^8) + b(2^4) + c$

$0 = 256a + 16b + c$ .... [4]

Subtract [1] from [4] and simplify

$-6 = 17a + b$ .... [5]

Subtract [3] from [5]

$a = 2$

Substituting $a = 2$ into [3] gives $b = -40$

Substituting $a = 2$, $b = -40$ into [1] gives $c = 128$

So, $a = 2$, $b = -40$, $c = 128$

EXERCISE 1F

1 **a** $y = x^2 - 6x + 8$ is a parabola

Comparing $y = x^2 - 6x + 8$ with $y = ax^2 + bx + c$

The value of $a = 1$ so $a > 0$ which means the parabola is a $\cup$ shape.

The x intercepts are found by substituting $y = 0$ into:

$y = x^2 - 6x + 8$

$0 = x^2 - 6x + 8$

$0 = (x - 2)(x - 4)$

$x = 2$ or $x = 4$

The x intercepts are at (2, 0) and (4, 0).

The y intercept is found by substituting $x = 0$ into

$y = x^2 - 6x + 8$

$y = 8$

The axes crossing points are (0, 8), (2, 0) and (4, 0)

The curve has a minimum (or lowest) point which is located at the vertex.

There is a line of symmetry which passes midway between $x = 2$ and $x = 4$, also passes through the vertex.

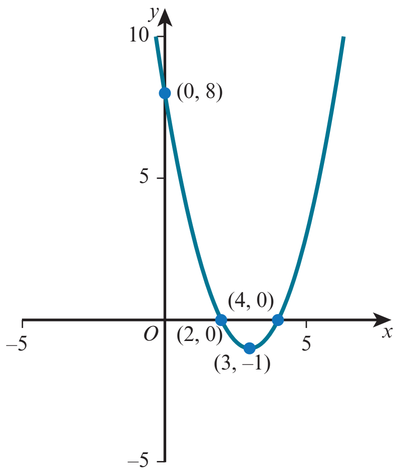
Its equation is $x = 3$

Substituting $x = 3$ into $y = x^2 - 6x + 8$ gives

$y = 3^2 - 6(3) + 8$

$y = -1$

The vertex (minimum point) is at (3, –1)



d $y = 12 + x - x^2$ is a parabola

Comparing $y = 12 + x - x^2$ with $y = ax^2 + bx + c$

The value of $a = -1$ so $a < 0$, which means the parabola is an $\cap$ shape.

The x-intercepts are found by substituting $y = 0$ into

$y = 12 + x - x^2$
$0 = 12 + x - x^2$
$0 = (3 + x)(4 - x)$
$x = -3$ or $x = 4$

The x intercepts are at (–3, 0) and (4, 0).

The y-intercept is found by substituting $x = 0$ into:

$y = 12 + x - x^2$
$y = 12$

Axes crossing points are (0, 12), (–3, 0) and (4, 0)

The curve has a maximum (or highest) point which is located at the vertex.

There is a line of symmetry which passes midway between $x = -3$ and $x = 4$ and also passes through the vertex.

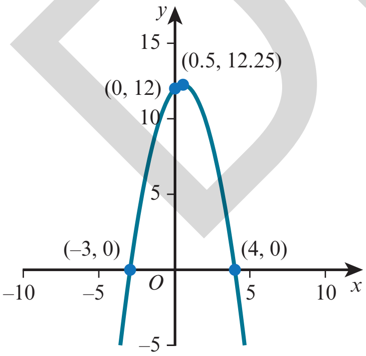
Its equation is $x = \frac{1}{2}$

Substituting $x = \frac{1}{2}$ into $y = 12 + x - x^2$ gives

$y = 12 + \frac{1}{2} - \left(\frac{1}{2}\right)^2$

$y = 12\frac{1}{4}$

The vertex (maximum point) is at $\left(\frac{1}{2}, 12\frac{1}{4}\right)$



2 **a** $2x^2 - 8x + 5$
$2(x^2 - 4x) + 5$
$2[(x - 2)^2 - 2^2] + 5$
$[2(x - 2)^2 - 8] + 5$
$2(x - 2)^2 - 3$

b The line of symmetry of the graph passes through the vertex which is at (2, –3).

Line of symmetry is $x = 2$.

3 **a** $y = 7 + 5x - x^2$
$y = 7 - (x^2 - 5x)$
$y = 7 - \left[\left(x - \frac{5}{2}\right)^2 - \frac{25}{4}\right]$
$y = \frac{53}{4} - \left(x - \frac{5}{2}\right)^2$

b Its graph is a ∩ shape.

The curve has a **maximum** (or highest) point i.e. a turning point which is located at the vertex $\left(\frac{5}{2}, \frac{53}{4}\right)$

The maximum point of the curve is at $\left(\frac{5}{2}, \frac{53}{4}\right)$ or $\left(2\frac{1}{2}, 13\frac{1}{4}\right)$

5 $x^2 - 7x + 8$

We are asked for the minimum value in this question. There are two methods which you can use:
Method 1 factorisation (if possible)
Method 2 completing the square

$x^2 - 7x + 8$ does not factorise so:

Completing the square gives:

$\left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + 8$

$\left(x - \frac{7}{2}\right)^2 - \frac{17}{4}$

Be careful! Here you are asked for the minimum value, not the minimum point.

Minimum value is $-4\frac{1}{4}$ [when $x = 3\frac{1}{2}$]

7 $y = 4x^2 + 2x + 5$ is a ∪ shaped parabola.
Complete the square to find the vertex (minimum point).

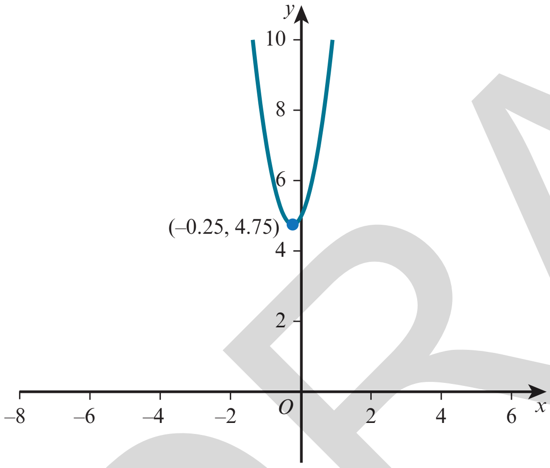
$$y = 4\left(x^2 + \frac{1}{2}x\right) + 5$$

$$y = 4\left[\left(x + \frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right] + 5$$

$$y = \left[4\left(x + \frac{1}{4}\right)^2 - \frac{1}{4}\right] + 5$$

$$y = 4\left(x + \frac{1}{4}\right)^2 + \frac{19}{4}$$

The vertex is at $\left(-\frac{1}{4}, \frac{19}{4}\right)$, which is above the x axis.



8 **Graph A** has its vertex at (4, 2). The point (6, 6) lies on the curve.

There are no x-intercepts.

There are three forms of a quadratic equation:

1 $\boldsymbol{y = ax^2 + bx + c}$
Any three different coordinate points on the parabola enables three equations to be formed and solved simultaneously. However, this is a long method and can be prone to calculation errors.

2 $\boldsymbol{y = a(x - d)(x - e)}$
To use this form you need to know the location of the x-intercepts (if any) need to be known.

3 $\boldsymbol{y = a(x - f)^2 + g}$
To use this form, the location of the vertex (f, g) needs to be known, plus one additional point on the parabola.

Using $y = a(x - f)^2 + g$ and substituting
$f = 4,\ g = 2$
$y = a(x - 4)^2 + 2$
Now substituting $x = 6,\ y = 6$ gives
$6 = a(6 - 4)^2 + 2$
$a = 1$
So, $y = (x - 4)^2 + 2$

Graph B The vertex is at (–2, –6).

The x intercepts are not clear.

The point (0, 10) lies on the curve.

Using $y = a(x - f)^2 + g$ and substituting
$f = -2,\ g = -6$
$y = a(x - -2)^2 - 6$
$y = a(x + 2)^2 - 6$
Now substituting $x = 0,\ y = 10$ gives
$10 = a(0 + 2)^2 - 6$
$a = 4$
So, $y = 4(x + 2)^2 - 6$

Graph C There are more than three pieces of information which can be read off the graph.

e.g. the vertex is at (2, 8).

The x intercepts are $x = -2,\ x = 6$

The point (0, 6) lies on the curve etc.

Using $y = a(x - d)(x - e)$

Substituting $d = -2,\ e = 6$
$y = a(x + 2)(x - 6)$
Now substituting $x = 2,\ y = 8$ gives
$8 = a(2 + 2)(2 - 6)$
$a = -\frac{1}{2}$
So, $y = -\frac{1}{2}(x + 2)(x - 6)$

9 $\mathbf{y = x^2 - 6x + 13}$

The graph is a ∪ shaped parabola

Completing the square gives:

$y = (x - 3)^2 + 4$

The vertex is at $(3, 4)$

$y = x^2 - 6x + 13$ is **A**

$\mathbf{y = -x^2 - 6x - 5}$

The graph is an ∩ shaped parabola

Completing the square gives:

$y = -(x^2 + 6x) - 5$

$y = -[(x + 3)^2 - 9] - 5$

$y = -(x + 3)^2 + 4$

The vertex is at $(-3, 4)$

$y = -x^2 - 6x - 5$ is G

> $y = -x^2 - bx - c$ is a reflection of $y = x^2 + bx + c$ in the x-axis, i.e. $f(x) \to -f(x)$
> $y = x^2 - bx + c$ is a reflection of $y = x^2 + bx + c$ in the y-axis, i.e. $f(x) \to -f(-x)$
> You will meet this again in Chapter 2.

Graph F is $y = x^2 + 6x + 5$ as it is a reflection of G in the x-axis

Graph D is $y = -x^2 + 6x - 13$ as it is a reflection of A in the x-axis

Graph E is $y = x^2 + 6x + 13$ as it is a reflection of A in the y-axis

Graph B is $y = x^2 - 6x + 5$ as it is reflection of F in the y-axis

Graph C is $y = -x^2 + 6x - 5$ as it is a reflection of G in the y-axis

Graph H is $y = -x^2 - 6x - 13$ as it is a reflection of E in the x-axis

(There are other ways to reach these solutions.)

10 Using $y = a(x - d)(x - e)$

Substituting $x = -2$ and $x = 4$ gives:

$y = a(x - -2)(x - 4)$

Substituting $x = 0,\ y = -24$ gives:

$-24 = a(0 - -2)(0 - 4)$

$a = 3$

Equation is $y = 3(x + 2)(x - 4)$ or

$y = 3x^2 - 6x - 24$

11 We do not know the x-intercepts nor the coordinates of the vertex.

We form three equations by substituting the three given coordinates into

$y = ax^2 + bx + c$ and solve them simultaneously.

Substituting $(-2, -3)$ gives $-3 = a(-2)^2 + b(-2) + c$ or $-3 = 4a - 2b + c$.......[1]

Substituting $(2, 9)$ gives $9 = a(2)^2 + b(2) + c$ or $9 = 4a + 2b + c$............[2]

Substituting $(6, 5)$ gives $5 = a(6)^2 + b(6) + c$ or $5 = 36a + 6b + c$..........[3]

[1] – [2] gives $-12 = -4b$ so $b = 3$

[2] – [3] gives $4 = -32a - 4b$

As $b = 3$, $4 = -32a - 12$ so $a = -\frac{1}{2}$

Substituting $a = -\frac{1}{2}$ and $b = 3$ into [1] gives:

$-3 = -2 - 6 + c$ so $c = 5$

The equation is $y = 5 + 3x - \frac{1}{2}x^2$

12 Using $y = a(x - f)^2 + g$

The vertex is at (p, q).

Substituting $f = p$ and $g = q$ gives:

$y = a(x - p)^2 + q$

Expanding gives:

$y = a(x^2 - 2px + p^2) + q$

$y = ax^2 - 2apx + ap^2 + q$ Proved

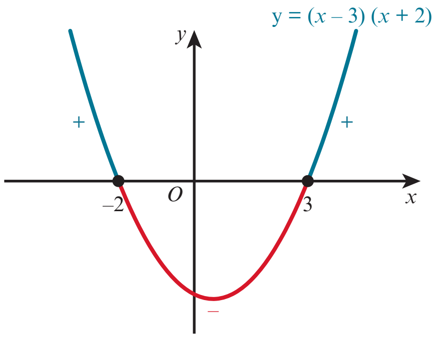
EXERCISE 1G

1 **b** $(x-3)(x+2) > 0$

Sketch the graph of $y = (x-3)(x+2)$

The graph is a ∪ shaped parabola.

The x-intercepts are at $x = -2$ and $x = 3$.



For $(x-3)(x+2) > 0$ we need to find the range of values of x for which the curve is positive (above the x axis).

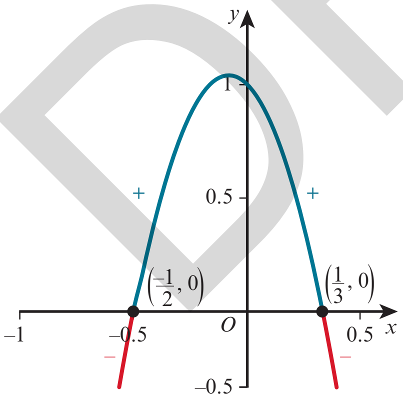
The solution is $x < -2$ or $x > 3$

f $(1-3x)(2x+1) < 0$

Sketch the graph of $y = (1-3x)(2x+1)$

The sketch is an ∩ shaped parabola.

The x-intercepts are at $x = -\frac{1}{2}$ and $x = \frac{1}{3}$.



For $(1-3x)(2x+1) < 0$ we need to find the range of values of x for which the curve is negative (below the x axis).

The solution is $x < -\frac{1}{2}$ or $x > \frac{1}{3}$

2 **a** $x^2 - 25 \geqslant 0$

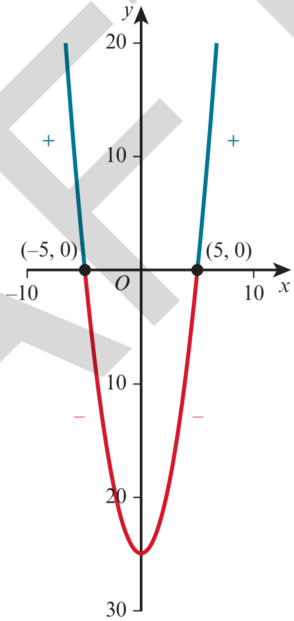
Factorising the left-hand side of the inequality:

$(x-5)(x+5) \geqslant 0$

Sketch the graph of $y = (x-5)(x+5)$

The sketch is an ∪ shaped parabola.

The x-intercepts are at $x = -5$ and $x = 5$.



For $x^2 - 25 \geqslant 0$ we need to find the range of values of x for which the curve is either zero or positive (on or above the x-axis).

The solution is $x \leqslant -5$ or $x \geqslant 5$.

e $6x^2 - 23x + 20 < 0$

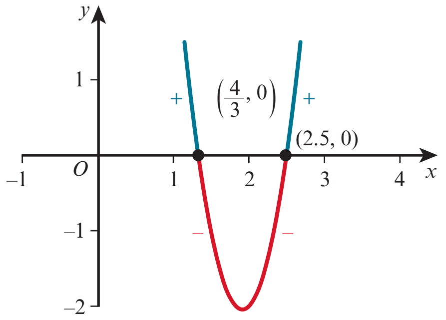
Factorising the left-hand side of the inequality:

$(3x-4)(2x-5) < 0$

Sketch the graph of $y = (3x-4)(2x-5)$

The sketch is a ∪ shaped parabola.

The x-intercepts are at $x = \frac{4}{3}$ and $x = \frac{5}{2}$.



For $6x^2 - 23x + 20 < 0$ we need to find the range of values of x for which the curve is negative (below the x-axis).

The solution is $\frac{4}{3} < x < \frac{5}{2}$

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3 **b** $15x < x^2 + 56$

Rearrange to give:

$x^2 - 15x + 56 > 0$

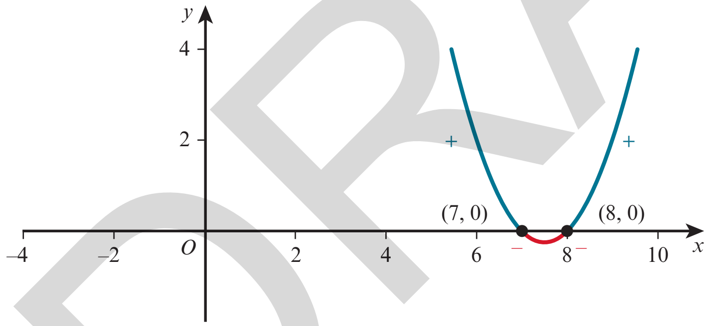
Factorising the left-hand side of the inequality:

$(x - 7)(x - 8) > 0$

Sketch the graph of $y = (x - 7)(x - 8)$

The sketch is a ∪ shaped parabola.

The x-intercepts are at $x = 7$ and $x = 8$



For $x^2 - 15x + 56 > 0$ we need to find the range of values of x for which the curve is positive (above the x-axis).

The solution is $x < 7$ or $x > 8$

g $(x + 4)^2 \geqslant 25$

Expand brackets and rearrange:

$x^2 + 8x - 9 \geqslant 0$

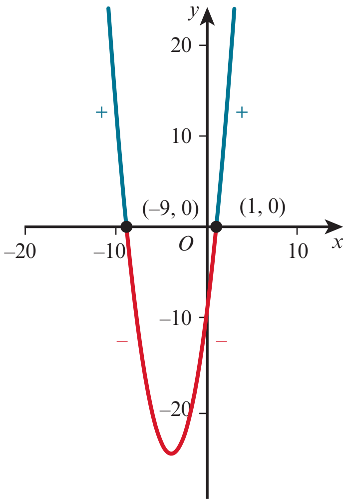
Factorising the left-hand side of the inequality:

$(x - 1)(x + 9) \geqslant 0$

Sketch the graph of $y = (x - 1)(x + 9)$

The sketch is a ∪ shaped parabola.

The x-intercepts are at $x = 1$ and $x = -9$



For $x^2 + 8x - 9 \geqslant 0$, we need to find the range of values of x for which the curve is either zero or positive (on or above the x-axis).

The solution is $x \leqslant -9$ or $x \geqslant 1$

4 $\dfrac{5}{2x^2 + x - 15} < 0$

$\dfrac{\text{positive value}}{\text{negative value}} < 0$ (the numerator here is always positive)

Factorising the denominator gives:

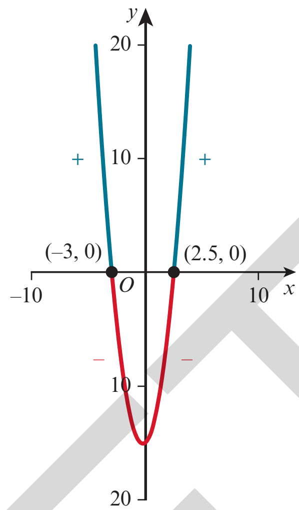
$\dfrac{5}{(2x-5)(x+3)} < 0$

5 is a positive value, so we need to find values of x

which make $(2x-5)(x+3)$ negative i.e. < 0

so, $(2x-5)(x+3) < 0$

A sketch of $y = (2x-5)(x+3)$, is a ∪ shaped parabola.



The x intercepts are at $x = 2.5$ and $x = -3$ (found when solving $2x - 5 = 0$ and $x + 3 = 0$)

We want $(2x-5)(x+3) < 0$ so, we need to find the range of values of x for which the curve is negative (below the x-axis).

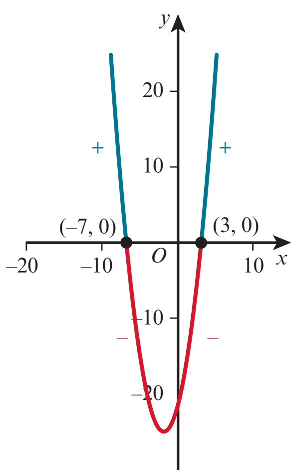
The solution is $-3 < x < 2.5$

5 b $\boldsymbol{x^2 + 4x - 21 \leqslant 0}$

Factorising the left-hand side of the inequality:

$(x+7)(x-3) \leqslant 0$

A sketch of $y = (x+7)(x-3)$ is a ∪ shaped parabola.



The x-intercepts are at $x = -7$ and $x = 3$

For $x^2 + 4x - 21 \leqslant 0$ we need to find the range of values of x for which the curve is either zero or negative (on or below the x axis)

The solution is $-7 \leqslant x \leqslant 3$

$\mathbf{x^2 - 9x + 8 > 0}$

Factorising the left-hand side of the inequality:

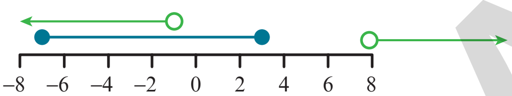
$(x - 1)(x - 8) > 0$

The graph of $y = (x - 1)(x - 8)$ is a $\cup$ shaped parabola.

The x-intercepts are at $x = 1$ and $x = 8$

For $x^2 - 9x + 8 > 0$ we need to find the range of values of x for which the curve is positive (above the x-axis).

The solution is $x < 1$ or $x > 8$



The diagram shows both solutions to be true when $-7 \leqslant x < 1$

6 $2^{x^2 - 3x - 40} > 1$

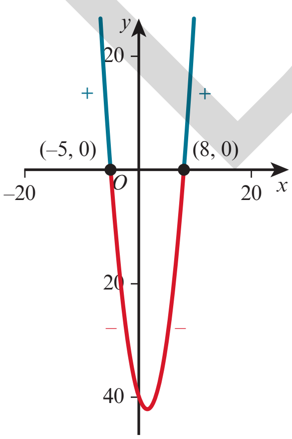
Since $2^0 = 1$, and $2^{\text{positive number}} > 1$

We need to solve $x^2 - 3x - 40 > 0$

Factorising the left-hand side of the inequality:

$(x + 5)(x - 8) > 0$

The sketch of $y = (x + 5)(x - 8)$ is a $\cup$ shaped parabola.



The x-intercepts are at $x = -5$ and $x = 8$

For $x^2 - 3x - 40 > 0$ we need to find the range of values of x for which the curve is positive (above the x axis).

The solution is $x < -5$ or $x > 8$

E 7 a $\dfrac{x}{x-1} \geqslant 3$

Rearrange $\dfrac{x}{x-1} - 3 \geqslant 0$

Write as a single fraction on the left-hand side:

$$\frac{x}{x-1} - \frac{3(x-1)}{x-1} \geqslant 0$$

$$\frac{x - 3(x-1)}{x-1} \geqslant 0$$

$$\frac{3 - 2x}{x-1} \geqslant 0$$

Find the values of x which each make the numerator and the denominator zero.

i.e. $3 - 2x = 0$ so $x = \dfrac{3}{2}$

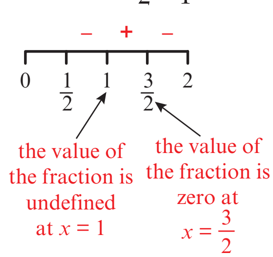
$x - 1 = 0$ so $x = 1$ (if the denominator of a fraction is zero then its value is undefined)

Use a number line to test numbers around $x = \dfrac{3}{2}$ and $x = 1$

If $x = 0$ then substituting into $\dfrac{3-2x}{x-1}$ becomes $\dfrac{3-2(0)}{0-1}$ which is negative.

If $x = 1.25$ then substituting into $\dfrac{3-2x}{x-1}$ becomes $\dfrac{3-2(1.25)}{1.25-1}$ which is positive.

If $x = 2$ then substituting into $\dfrac{3-2x}{x-1}$ becomes $\dfrac{3-2(2)}{2-1}$ which is negative.



$\dfrac{x}{x-1} \geqslant 3$ for values of x which satisfy:

$1 < x \leqslant \dfrac{3}{2}$

b $\dfrac{x(x-1)}{x+1} > x$

Rearrange $\dfrac{x(x-1)}{x+1} - x > 0$

Write as a single fraction on the left-hand side:

$$\frac{x(x-1)}{x+1} - \frac{x(x+1)}{x+1} > 0$$

$$\frac{x(x-1) - x(x+1)}{x+1} > 0$$

$$\frac{x^2 - x - x^2 - x}{x+1} > 0$$

$$\frac{-2x}{x+1} > 0$$

Find the values of x which each make the numerator and the denominator zero.

i.e. $-2x = 0$ so $x = 0$

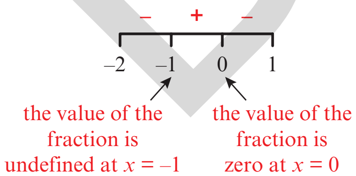
$x + 1 = 0$ so $x = -1$ (if the denominator of a fraction is zero then its value is undefined)

Use a number line to test numbers around $x = 0$ and $x = -1$

If $x = -2$ then substituting into $\dfrac{-2x}{x+1}$ becomes $\dfrac{-2(-2)}{-2+1}$ which is negative.

If $x = -\dfrac{1}{2}$ then substituting into $\dfrac{-2x}{x+1}$ becomes $\dfrac{-2\left(-\frac{1}{2}\right)}{-\frac{1}{2}+1}$ which is positive.

If $x = 1$ then substituting into $\dfrac{-2x}{x+1}$ becomes $\dfrac{-2(1)}{1+1}$ which is negative.



$\dfrac{x(x-1)}{x+1} > x$ for values of x which satisfy $-1 < x < 0$

c $\dfrac{x^2 - 9}{x-1} \geqslant 4$

Rearrange $\dfrac{x^2 - 9}{x-1} - 4 \geqslant 0$

Write as a single fraction on the left-hand side:

$$\frac{x^2 - 9}{x-1} - \frac{4(x-1)}{x-1} \geqslant 0$$

$$\frac{x^2 - 9 - 4(x-1)}{x-1} \geqslant 0$$

$$\frac{x^2 - 9 - 4x + 4}{x-1} \geqslant 0$$

$$\frac{x^2 - 4x - 5}{x-1} \geqslant 0$$

Find the values of x which each make the numerator and the denominator zero.

i.e. $x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

so $x = 5$ or $x = -1$

$x - 1 = 0$ so $x = 1$ (if the denominator of a fraction is zero then its value is undefined).

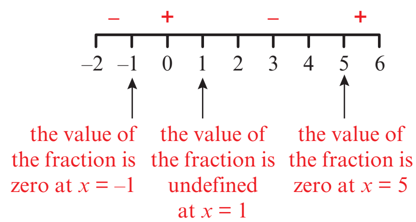
Use a number line to test numbers around $x = -1$, $x = 5$ and $x = 1$

If $x = -2$ then $\dfrac{x^2 - 4x - 5}{x-1}$ becomes $\dfrac{(-2)^2 - 4(-2) - 5}{-2-1}$ which is negative

If $x = 0$ then $\dfrac{x^2 - 4x - 5}{x-1}$ becomes $\dfrac{(0)^2 - 4(0) - 5}{0-1}$ which is positive

If $x = 2$ then $\dfrac{x^2 - 4x - 5}{x-1}$ becomes $\dfrac{(2)^2 - 4(2) - 5}{2-1}$ which is negative

If $x = 6$ then $\dfrac{x^2 - 4x - 5}{x-1}$ becomes $\dfrac{(6)^2 - 4(6) - 5}{6-1}$ which is positive



$\dfrac{x^2-9}{x-1} \geqslant 4$ for values of x which satisfy:

$-1 \leqslant x < 1$ or $x \geqslant 5$

d $\dfrac{x^2-2x-15}{x-2} \geqslant 0$

$\dfrac{(x-5)(x+3)}{x-2} \geqslant 0$

Find the values of x which each make the numerator and the denominator zero.

For the numerator, solve $(x-5)(x+3)=0$

so $x=5$ or $x=-3$

For the denominator, solve $x-2=0$

so $x=2$ (if the denominator of a fraction is zero then its value is undefined)

Use a number line to test numbers around $x=-3$, $x=2$ and $x=5$

If $x=-4$ then $\dfrac{(x-5)(x+3)}{x-2}$

becomes $\dfrac{(-4-5)(-4+3)}{-4-2}$ which is negative.

If $x=0$ then $\dfrac{(x-5)(x+3)}{x-2}$ becomes

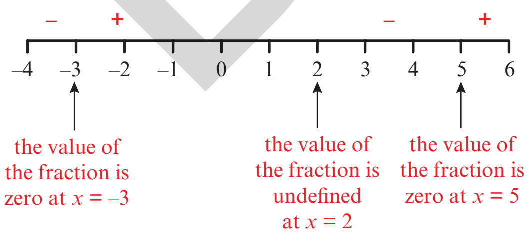
$\dfrac{(0-5)(0+3)}{0-2}$ which is positive.

If $x=3$ then $\dfrac{(x-5)(x+3)}{x-2}$ becomes

$\dfrac{(3-5)(3+3)}{3-2}$ which is negative.

If $x=6$ then $\dfrac{(x-5)(x+3)}{x-2}$ becomes

$\dfrac{(6-5)(6+3)}{6-2}$ which is positive.



$\dfrac{x^2-2x-15}{x-2} \geqslant 0$ for values of x which satisfy:

$-3 \leqslant x < 2$ or $x \geqslant 5$

e $\dfrac{x^2+4x-5}{x^2-4} \leqslant 0$

$\dfrac{(x+5)(x-1)}{(x-2)(x+2)} \leqslant 0$

Find the values of x which each make the numerator and the denominator zero.

For the numerator, solve $(x+5)(x-1)=0$

so $x=-5$ or $x=1$

For the denominator, solve $(x-2)(x+2)=0$

so $x=2$ or $x=-2$ (if the denominator of a fraction is zero then its value is undefined).

Use a number line to test numbers around $x=-5$, $x=-2$, $x=1$ and $x=2$

If $x=-6$ then substituting into $\dfrac{(x+5)(x-1)}{(x-2)(x+2)}$

becomes $\dfrac{(-6+5)(-6-1)}{(-6-2)(-6+2)}$ which is positive.

If $x=-3$ then substituting into $\dfrac{(x+5)(x-1)}{(x-2)(x+2)}$

becomes $\dfrac{(-3+5)(-3-1)}{(-3-2)(-3+2)}$ which is negative.

If $x=0$ then substituting into $\dfrac{(x+5)(x-1)}{(x-2)(x+2)}$

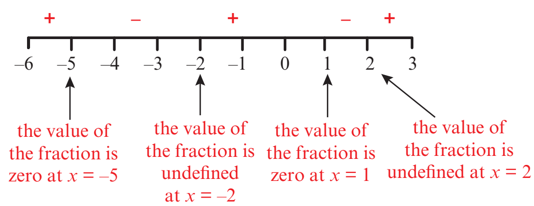
becomes $\dfrac{(0+5)(0-1)}{(0-2)(0+2)}$ which is positive.

If $x=1.5$ then substituting into $\dfrac{(x+5)(x-1)}{(x-2)(x+2)}$

becomes $\dfrac{(1.5+5)(1.5-1)}{(1.5-2)(1.5+2)}$ which is negative.

If $x=3$ then substituting into $\dfrac{(x+5)(x-1)}{(x-2)(x+2)}$

becomes $\dfrac{(3+5)(3-1)}{(3-2)(3+2)}$ which is positive.



$\dfrac{x^2+4x-5}{x^2-4} \leqslant 0$ for values of x which satisfy:

$-5 \leqslant x < -2$ or $1 \leqslant x < 2$

f $\dfrac{x-3}{x+4} \geqslant \dfrac{x+2}{x-5}$

Rearrange $\dfrac{x-3}{x+4} - \dfrac{x+2}{x-5} \geqslant 0$

Write as a single fraction on the left-hand side:

$$\frac{(x-3)(x-5)-(x+2)(x+4)}{(x+4)(x-5)} \geqslant 0$$

Be careful with the numerator!

$$\frac{x^2-8x+15-[x^2+6x+8]}{(x+4)(x-5)} \geqslant 0$$

$$\frac{x^2-8x+15-x^2-6x-8}{(x+4)(x-5)} \geqslant 0$$

$$\frac{7-14x}{(x+4)(x-5)} \geqslant 0$$

$$\frac{7(1-2x)}{(x+4)(x-5)} \geqslant 0$$

Find the values of x which each make the numerator and the denominator zero.

For the numerator, solve $7(1-2x)=0$

so $x=\dfrac{1}{2}$

For the denominator, solve $(x+4)(x-5)=0$

so $x=-4$ or $x=5$ (if the denominator of a fraction is zero then its value is undefined).

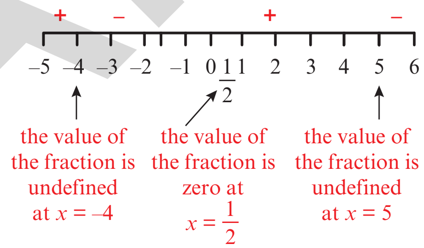
Use a number line to test numbers around $x=-4$, $x=\dfrac{1}{2}$ and $x=5$

If $x=-5$ then $\dfrac{7(1-2x)}{(x+4)(x-5)}$ becomes $\dfrac{7(1-2(-5))}{(-5+4)(-5-5)}$ which is positive.

If $x=0$ then $\dfrac{7(1-2x)}{(x+4)(x-5)}$ becomes $\dfrac{7(1-2(0))}{(0+4)(0-5)}$ which is negative.

If $x=1$ then $\dfrac{7(1-2x)}{(x+4)(x-5)}$ becomes $\dfrac{7(1-2(1))}{(1+4)(1-5)}$ which is positive.

If $x=6$ then $\dfrac{7(1-2x)}{(x+4)(x-5)}$ becomes $\dfrac{7(1-2(6))}{(6+4)(6-5)}$ which is negative.



$\dfrac{x-3}{x+4} \geqslant \dfrac{x+2}{x-5}$ for values of x which satisfy:

$x<-4$ or $\dfrac{1}{2} \leqslant x < 5$

EXERCISE 1H

1 **b** $x^2+5x-36=0$

$a=1,\ b=5,\ c=-36$

Substituting into b^2-4ac gives:

$5^2-4(1)(-36)$ which is >0 so there are two distinct real roots.

e $2x^2-7x+8=0$

$a=2,\ b=-7,\ c=8$

Substituting into b^2-4ac gives:

$(-7)^2-4(2)(8)$ which is <0 so there are no real roots.

2 $2-5x=\dfrac{4}{x}$

Rearrange and simplify:

$5x^2-2x+4=0$

$a=5,\ b=-2,\ c=4$

Substituting into b^2-4ac gives:

$(-2)^2-4(5)(4)$ which is <0 so there are no real roots.

3 $(x+5)(x-7)=0$ which expanded gives:

$x^2-2x-35=0$

So, $b=-2$ and $c=-35$

4 **b** $4x^2+4(k-2)x+k=0$

So, $a=4,\ b=4(k-2),\ c=k$

For two equal roots $b^2-4ac=0$

$[4(k-2)]^2-4(4)(k)=0$ which simplified gives:

$16k^2-80k+64=0$ or:

$k^2-5k+4=0$

$(k-1)(k-4)=0$

So, $k=1$ or $k=4$

e $(k+1)x^2+kx-2k=0$

$a=k+1,\ b=k,\ c=-2k$

For two equal roots $b^2-4ac=0$

$k^2-4(k+1)(-2k)=0$

$k^2+8k(k+1)=0$

$9k^2+8k=0$

$k(9k+8)=0$

$k=0$ or $k=-\dfrac{8}{9}$

5 **b** $2x^2-5x=4-k$

Rearranging gives: $2x^2-5x+(k-4)=0$

$a=2,\ b=-5,\ c=k-4$

For two distinct roots $b^2-4ac>0$

$(-5)^2-4(2)(k-4)>0$ which simplifies to:

$8k-57<0$

$k<\dfrac{57}{8}$

d $kx^2+2(k-1)x+k=0$

$a=k,\ b=2(k-1),\ c=k$

For two distinct roots $b^2-4ac>0$

$[2(k-1)]^2-4(k)(k)>0$

$-8k+4>0$

$k<\dfrac{1}{2}$

6 **b** $3x^2+5x+k+1=0$

$a=3,\ b=5,\ c=k+1$

For no real roots $b^2-4ac<0$

$5^2-4(3)(k+1)<0$

$-12k+13<0$

$k>\dfrac{13}{12}$

e $kx^2+2kx=4x-6$

$kx^2+(2k-4)x+6=0$

$a=k,\ b=2k-4,\ c=6$

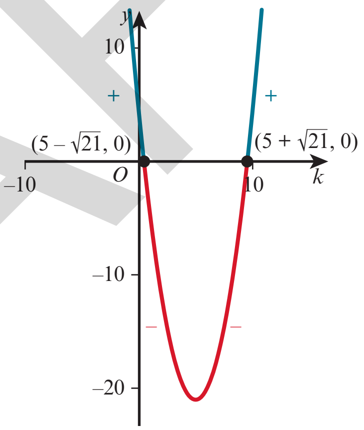
For no real roots $b^2-4ac<0$

$(2k-4)^2-4(k)(6)<0$ which simplifies to:

$4k^2-40k+16<0$ or

$k^2-10k+4<0$

The sketch of $y=k^2-10k+4$ is a ∪ shaped parabola.



$k^2-10k+4$ does not factorise so to find the k-intercepts we must use the quadratic formula.

$$k=\frac{-(-10)\pm\sqrt{(-10)^2-4\times1\times(4)}}{2\times1}$$

$k=\dfrac{10+\sqrt{84}}{2}$ or $k=\dfrac{10-\sqrt{84}}{2}$ which simplify to give:

$k=5+\sqrt{21}$ or $k=5-\sqrt{21}$

The k-intercepts are at $k=5+\sqrt{21}$ and $k=5-\sqrt{21}$

For $k^2 - 10k + 4 < 0$ we need to find the range of values of k for which the curve is negative (below the k axis).

The solution is $5 - \sqrt{21} < k < 5 + \sqrt{21}$

7 $kx^2 + px + 5 = 0$

$a = k, \ b = p, \ c = 5$

For repeated real roots $b^2 - 4ac = 0$

$p^2 - 4(k)(5) = 0$

$k = \dfrac{p^2}{20}$

8 $kx^2 - 5x + 2 = 0$

$a = k, \ b = -5, \ c = 2$

For real roots $b^2 - 4ac \geqslant 0$

$(-5)^2 - 4(k)(2) \geqslant 0$

$k \leqslant \dfrac{25}{8}$

9 $2kx^2 + 5x - k = 0$

$a = 2k, \ b = 5, \ c = -k$

$b^2 - 4ac$ is $5^2 - 4(2k)(-k)$ which simplifies to: $25 + 8k^2$

$25 + 8k^2 \geqslant 25$ for all values of k i.e. it is always positive

So, $b^2 - 4ac > 0$ which proves that the roots are real and distinct for all real values of k.

10 $x^2 + (k - 2)x - 2k = 0$

$a = 1, \ b = k - 2, \ c = -2k$

$b^2 - 4ac$ is $(k - 2)^2 - 4(1)(-2k)$

which simplifies to $k^2 + 4k + 4$ or $(k + 2)^2$

$(k + 2)^2$ is always $\geqslant 0$

Therefore the roots are real for all values of k.

11 $x^2 + kx + 2 = 0$

$a = 1, \ b = k, \ c = 2$

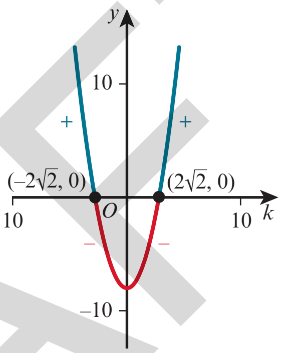
For real roots $b^2 - 4ac \geqslant 0$

So, $k^2 - 4(1)(2) \geqslant 0$ or $k^2 - 8 \geqslant 0$

Factorising the left-hand side of the inequality gives:

$\left(k - \sqrt{8}\right)\left(k + \sqrt{8}\right) \geqslant 0$

The sketch of $y = \left(k - \sqrt{8}\right)\left(k + \sqrt{8}\right)$ is a ∪ shaped parabola.



The k-intercepts are $k = 2\sqrt{2}$ and $k = -2\sqrt{2}$

we need to find the range of values of k for which the curve is either zero or positive (on or above the k axis).

The solution is $k \leqslant -2\sqrt{2}$ or $k \geqslant 2\sqrt{2}$

Therefore the equation has real roots if $k \geqslant 2\sqrt{2}$,

the other values of k are $k \leqslant -2\sqrt{2}$.

EXERCISE 1I

1 If $y = kx + 1$ is a tangent to $y = x^2 - 7x + 2$ then there should only be one solution to the equation formed by solving $y = kx + 1$ and $y = x^2 - 7x + 2$ simultaneously.

$x^2 - 7x + 2 = kx + 1$ when rearranged gives:

$x^2 - (7 + k)x + 1 = 0$

$a = 1, \ b = -(7 + k), \ c = 1$

For one repeated real root $b^2 - 4ac = 0$

$[-(7+k)]^2 - 4(1)(1) = 0$

$k^2 + 14k + 45 = 0$

$(k+5)(k+9) = 0$

$k = -5$ or $k = -9$

2 The x-axis has the equation $y = 0$

If $y = 0$ is a tangent to $y = x^2 - (k+3)x + (3k+4)$ then there should only be one solution to the equation formed by solving $y = 0$ and $y = x^2 - (k+3)x + (3k+4)$ simultaneously.

$x^2 - (k+3)x + (3k+4) = 0$

$a = 1,\ b = -(k+3),\ c = (3k+4)$

For one repeated real root $b^2 - 4ac = 0$

$[-(k+3)]^2 - 4(1)(3k+4) = 0$

$k^2 - 6k - 7 = 0$

$(k+1)(k-7) = 0$

$k = -1$ or $k = 7$

3 If $x + ky = 12$ is a tangent to $y = \frac{5}{x-2}$ then there should only be one solution to the equation formed by solving $x + ky = 12$ ....[1] and $y = \frac{5}{x-2}$ ....[2] simultaneously.

From [1] $y = \frac{12-x}{k}$ and substituting for y in [2] gives:

$\frac{12-x}{k} = \frac{5}{x-2}$

Simplifying and rearranging gives:

$x^2 - 14x + (5k+24) = 0$

$a = 1,\ b = -14,\ c = (5k+24)$

For one repeated real root $b^2 - 4ac = 0$

$(-14)^2 - 4(1)(5k+24) = 0$

$k = 5$

4 **a** If $y = k - 3x$ is a tangent to $0 = x^2 + 2xy - 20$ then there should only be one solution to the equation formed by solving $y = k - 3x$ ......[1] and $x^2 + 2xy - 20 = 0$ .....[2] simultaneously.

Substituting for y in [2] gives:

If $x^2 + 2x(k-3x) - 20 = 0$

Rearranging this equation gives:

$5x^2 - 2kx + 20 = 0$

$a = 5,\ b = -2k,\ c = 20$

As $b^2 - 4ac = 0$ for one repeated root

$(-2k)^2 - 4(5)(20) = 0$

$k = \pm 10$

b First, substitute $k = -10$ into $y = k - 3x$ giving:

$y = -10 - 3x$

And then as $x^2 + 2xy - 20 = 0$ solving these two equations simultaneously gives:

$x^2 + 2x(-10 - 3x) - 20 = 0$

$-5x^2 - 20x - 20 = 0$

This simplifies to:

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

Substituting $x = -2$ into $y = -10 - 3x$ gives:

$y = -4$

Second, substitute $k = 10$ into $y = k - 3x$ giving:

$y = 10 - 3x$

And then as $x^2 + 2xy - 20 = 0$ solving these two equations simultaneously gives:

$x^2 + 2x(10 - 3x) - 20 = 0$ and then simplifies to:

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

Substituting $x = 2$ into $y = 10 - 3x$ gives:

$y = 4$

The coordinates are $(2, 4)$ and $(-2, -4)$

6 $y = 2x - 1$ ....[1]

$y = x^2 + kx + 3$ ....[2]

Substitute for y in [2]

$2x - 1 = x^2 + kx + 3$

Rearrange:

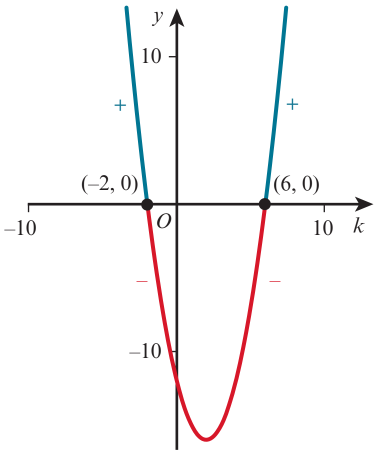
$x^2 + (k-2)x + 4 = 0$

$a = 1,\ b = k - 2,\ c = 4$

For two distinct roots $b^2 - 4ac > 0$

$(k-2)^2 - 4(1)(4) > 0$
$k^2 - 4k - 12 > 0$
$(k-6)(k+2) > 0$

A sketch of $y = (k-6)(k+2)$, is a ∪ shaped parabola.



The k–intercepts are at $k = -2$ and $k = 6$

For $k^2 - 4k - 12 > 0$ we need to find the range of values of k for which the curve is positive (above the k-axis).

The solution is $k < -2$ and $k > 6$.

9 $y = mx + 5$ ....[1]
$y = x^2 - x + 6$ ....[2]

Substitute for y in [2]

$mx + 5 = x^2 - x + 6$

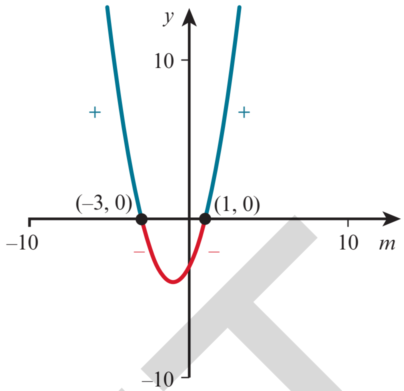
Rearrange:

$x^2 - (1 + m)x + 1 = 0$
$a = 1,\ b = -(1 + m),\ c = 1$

If the straight line does not meet the curve, then there are no real solutions to the equation.

So, $b^2 - 4ac < 0$
$[-(1 + m)]^2 - 4(1)(1) < 0$
$m^2 + 2m - 3 < 0$
$(m-1)(m+3) < 0$

A sketch of $y = (m-1)(m+3)$ is a ∪ shaped parabola.



The m-intercepts are at $m = -3$ and $m = 1$

For $m^2 + 2m - 3 < 0$ we need to find the range of values of m for which the curve is negative (below the m-axis).

The solution is $-3 < m < 1$

11 $y = kx + 6$ ....[1]
$x^2 + y^2 - 10x + 8y = 84$ ....[2]

Substitute for y in [2]

$x^2 + (kx + 6)^2 - 10x + 8(kx + 6) = 84$

Simplified and rearranged:

$(1 + k^2)x^2 + (20k - 10)x = 0$

If the straight line is a tangent to the curve, then this equation has one root

so $b^2 - 4ac = 0$

$a = 1 + k^2,\ b = 20k - 10,\ c = 0$

$(20k - 10)^2 - 4(1 + k^2)(0) = 0$

$(20k - 10)^2 = 0$

$k = \frac{1}{2}$

12 $y = mx + c$ ....[1]
$y = x^2 - 4x + 4$ ....[2]

If the line is a tangent to the curve then there should be one solution to the equation

$mx + c = x^2 - 4x + 4$

Rearranged:

$x^2 - (4 + m)x + (4 - c) = 0$

$a = 1,\ b = -(4 + m),\ c = (4 - c)$

For one (repeated) root $b^2 - 4ac = 0$

$[-(4 + m)]^2 - 4(1)(4 - c) = 0$

$16 + 8m + m^2 - 16 + 4c = 0$

$m^2 + 8m + 4c = 0$ proved.

13 $y = mx + c$ ....[1]

$ax^2 + by^2 = c$ ....[2]

Substitute for y in [2]

$ax^2 + b(mx + c)^2 = c$

Expanded gives:

$ax^2 + bm^2x^2 + (2bcm)x + bc^2 - c = 0$

$(a + bm^2)x^2 + (2bcm)x + (bc^2 - c) = 0$*

If a line is a tangent to the curve then an equation of the form:

$ax^2 + bx + c = 0$ should have one solution.

i.e. $b^2 - 4ac = 0$

For our equation*

$$(2bcm)^2 - 4(a + bm^2)(bc^2 - c) = 0$$

$$4b^2c^2m^2 - 4abc^2 + 4ac - 4b^2c^2m^2 + 4bm^2c = 0$$

$$-4abc^2 + 4ac + 4bm^2c = 0$$

$$4bm^2c = 4abc^2 - 4ac$$

$m^2 = \dfrac{4abc^2 - 4ac}{4bc}$ dividing each term by $4c$

$m^2 = \dfrac{abc - a}{b}$ Proved