CAMBRIDGE

Brighter Thinking

A Level Mathematics for AQA

Student Book 2 (Year 2)

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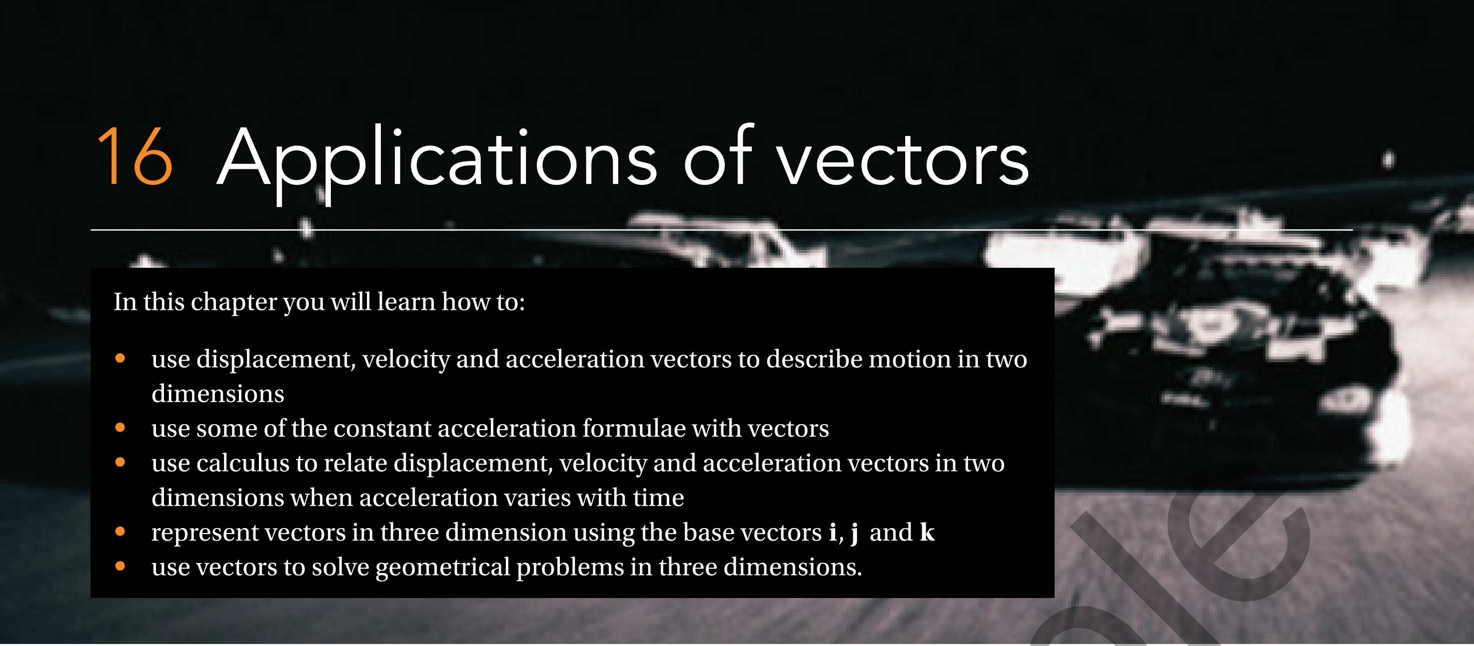
This resource has been entered into the AQA approval process.

Contents

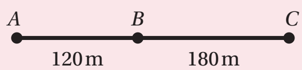


16 Applications of vectors

In this chapter you will learn how to:

- use displacement, velocity and acceleration vectors to describe motion in two dimensions
- use some of the constant acceleration formulae with vectors
- use calculus to relate displacement, velocity and acceleration vectors in two dimensions when acceleration varies with time
- represent vectors in three dimension using the base vectors **i**, **j** and **k**
- use vectors to solve geometrical problems in three dimensions.

Before you start…

Student Book 1	You should be able to link displacement vectors to coordinates and perform operations with vectors.	1 Consider the points $A(2, 5)$, $B(-1, 3)$ and $C(7, -2)$. Let $\mathbf{p} = \overrightarrow{AB}$ and $\mathbf{q} = \overrightarrow{BC}$. Write in column vector form: a $\mathbf{p}$ b $\mathbf{q} - \mathbf{p}$ c $4\mathbf{q}$ d $\overrightarrow{AC}$.
Student Book 1	You should be able to find the magnitude and direction of a vector.	2 Find the magnitude and direction of the vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.
Student Book 1	You should understand the concepts of displacement and distance, instantaneous and average velocity and speed and acceleration.	3 In the diagram, positive displacement is measured to the right.  A particle takes 3 seconds to travel from B to C and another 7 seconds to travel from C to A. Find: a the average velocity and b the average speed for the whole journey.
Student Book 1	You should be able to use calculus to work with displacement, velocity and acceleration in one dimension.	4 A particle moves in a straight line with the velocity $v = 2\,\mathrm{e}^{t} - t^2$. Find: a the acceleration when $t = 3$. b an expression for the displacement from the starting position.
Student Book 1	You should be able to use constant acceleration formulae in one dimension.	5 A particle accelerates uniformly from $3\ \mathrm{m\,s^{-1}}$ to $7\ \mathrm{m\,s^{-1}}$ while covering a distance of 60 m in a straight line. Find the acceleration.
Chapter 12	You should be able to work with curves defined parametrically.	6 Find the Cartesian equation of the curve with parametric equations $x = 1 - 2t^2$, $y = 1 + t$.

Why do you need to use vectors to describe motion?

In Student Book 1, you studied motion in a straight line. You saw how displacement, velocity and acceleration are related through differentiation and integration:

$$v = \frac{dx}{dt}, \quad x = \int v\,dt$$
$$a = \frac{dv}{dt}, \quad v = \int a\,dt$$

In the special case when the acceleration is constant, you can use the constant acceleration equations:

$$v = u + at, \quad v^2 = u^2 + 2as, \quad s = ut + \frac{1}{2}at^2, \quad s = vt - \frac{1}{2}at^2, \quad s = \frac{1}{2}(u+v)t$$

But the real world has three dimensions, and objects do not always move in a straight line. You need to be able to describe positions and motion in a plane (such as a car moving around a race track) or in space (for example flight paths of aeroplanes). This requires the use of vectors to describe displacement, velocity and acceleration.

Rewind

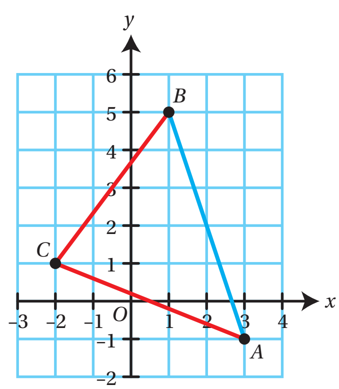
Vectors were introduced in Student Book 1. Remember that you can also write the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as $2\mathbf{i} + 3\mathbf{j}$.

Section 1: Describing motion in two dimensions

When a particle moves in two dimensions, the displacement, velocity and acceleration are vectors. The distance and speed are still scalars.

WORKED EXAMPLE 16.1

Points A, B and C have position vectors $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, where the distance is measured in metres. A particle travels in a straight line between each pair of points. It takes 5 seconds to travel from A to C and then a further 3 seconds to travel from C to B. Find:



a the average velocity and average speed from C to B
b the final displacement of the particle from A
c the average velocity for the whole journey
d the average speed for the whole journey.

a $\overrightarrow{CB} = \mathbf{b} - \mathbf{c}$

$= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

First find the displacement from C to B. This is the difference between the position vectors.

Remember: $\overrightarrow{CB} = \mathbf{b} - \mathbf{c}$

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Average velocity $= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \div 3$

$= \begin{pmatrix} 1 \\ 1.33 \end{pmatrix} \text{m s}^{-1}$

Then use average velocity $= \dfrac{\text{displacement}}{\text{time}}$.

$CB = \sqrt{3^2 + 4^2} = 5$

Use Pythagoras' theorem to find the distance CB.

Speed $= \dfrac{5}{3} = 1.67 \text{ m s}^{-1}$

Then use average speed $= \dfrac{\text{distance}}{\text{time}}$.

b $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

The displacement from A to B is the difference between their position vectors. It is highlighted blue in the diagram.

c Total time $= 5 + 3 = 8$ s

Average velocity $= \dfrac{\text{final displacement}}{\text{time}}$

Average velocity $= \begin{pmatrix} -2 \\ 6 \end{pmatrix} \div 8$

$= \begin{pmatrix} -0.25 \\ 0.75 \end{pmatrix} \text{m s}^{-1}$

d $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

To get the total distance travelled, add the distance from A to C to the distance from C to B. This is highlighted red in the diagram.

$AC = \sqrt{5^2 + 2^2} = \sqrt{29}$

$\overrightarrow{CB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$CB = \sqrt{3^2 + 4^2} = \sqrt{25}$

Total distance $= \sqrt{29} + \sqrt{25} = 10.4$ m

$\therefore$ average speed $= \dfrac{10.4}{8} = 1.3 \text{ m s}^{-1}$

Average speed $= \dfrac{\text{total distance}}{\text{time}}$

You have already found that the total time is 8 seconds.

You can calculate average acceleration by considering the change in velocity.

Common error

Notice that the average speed is **not** the magnitude of the average velocity vector. This is because the particle changes direction during the motion.

WORKED EXAMPLE 16.2

A particle moves in a plane. It passes point A with velocity $(3\mathbf{i} - 2\mathbf{j})\ \mathrm{m\,s^{-1}}$ and passes point B 4 seconds later with velocity $(2\mathbf{i} + 5\mathbf{j})\ \mathrm{m\,s^{-1}}$.

Find the magnitude and direction of the average acceleration of the particle between A and B.

Average acceleration $= ((2\underset{\sim}{i} + 5\underset{\sim}{j}) - (3\underset{\sim}{i} - 2\underset{\sim}{j})) \div 4$

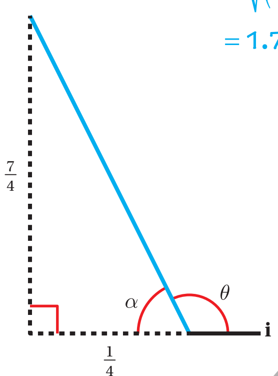
$$= \left(-\frac{1}{4}\underset{\sim}{i} + \frac{7}{4}\underset{\sim}{j}\right)\ \mathrm{m\,s^{-2}}$$

First find the acceleration vector using average acceleration $= \dfrac{\text{change in velocicty}}{\text{time}}$.

The magnitude is $|\underset{\sim}{a}| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{7}{4}\right)^2}$

$$= 1.77\ \mathrm{m\,s^2}$$

Then find its magnitude.



You can use any angle to define the direction as long as you clearly state where you are measuring the angle from. The angle above (or anti-clockwise from) the vector **i** is the most common choice. Draw a diagram to make sure you find the angle you intended!

$\tan\alpha = \dfrac{\frac{7}{4}}{\frac{1}{4}} = 7$

$\alpha = 81.7$

$\therefore \theta = 180 - 81.7$

$= 98.1°$

The direction is $98.1°$ above the vector $\underset{\sim}{i}$.

Acceleration may cause a change in the direction of the velocity as well as its magnitude (speed). This means that the object will not necessarily move in a straight line. If you know how the displacement vector varies with time, you can sometimes find the Cartesian equation of the object's path.

Tip

The path an object follows is also called a **trajectory**.

Rewind

The displacement vector gives the parametric equations of the object's path, with the parameter being time. See Chapter 12, Section 3 for a reminder of parametric equations.

WORKED EXAMPLE 16.3

A particle moves in a plane. At time t the particle is at a point and its displacement from the origin O is given by the position vector $\overrightarrow{OP} = \begin{pmatrix} t+2 \\ 1-t^2 \end{pmatrix}$.

a Prove that the particle moves along a parabola.
b Sketch the parabola.

a The coordinates of P are:
$x = t+2$, $y = 1-t^2$

The two components of the position vector give the x- and y-coordinates of the particle's position.

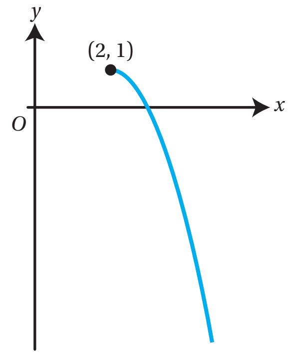
$t = x-2$

You want to find y in terms of x, so to eliminate t, substitute t from the first equation into the second equation.

$\therefore y = 1-(x-2)^2$
$y = -x^2+4x-3$

Hence the path of the particle is a parabola.

b



Note that, since $t > 0$, the object covers only a part of the parabola with $x > 2$.

You can also look at two particles moving in a plane and ask questions about the distance between them, and whether they ever meet. To do this you need to work with position vectors.

Key point 16.1

- If a particle starts at the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, its position vector at time t is $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.
- Two particles, A and B, meet if $\mathbf{r}_A = \mathbf{r}_B$ for the same value of t.

Rewind

In Worked example 16.4, part **b** is an example of proof by contradiction – see Chapter 1 for a reminder.

WORKED EXAMPLE 16.4

Two particles, A and B, move in the same plane. A starts from the origin and moves with constant velocity $\mathbf{v}_A = (3\mathbf{i}-2\mathbf{j})\text{ m s}^{-1}$.

a Write down the position vector of A in terms of t.

Particle B starts from the point with position vector $(\mathbf{i}-5\mathbf{j})$ m and moves with constant velocity $\mathbf{v}_B = (\mathbf{i}+3\mathbf{j})\text{ m s}^{-1}$.

b Prove that A and B never meet.
c Find the minimum distance between the two particles.

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a $\underset{\sim}{r}_A = (3\underset{\sim}{i} - 2\underset{\sim}{j})t$

$= 3t\underset{\sim}{i} - 2t\underset{\sim}{j}$

Use $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ with $\mathbf{r}_0 = 0$ as A starts at the origin.

b Position vector of B is

This time $\mathbf{r}_0 = \mathbf{i} - 5\mathbf{j}$.

$\underset{\sim}{r}_B = (\underset{\sim}{i} - 5\underset{\sim}{j}) + (\underset{\sim}{i} + 3\underset{\sim}{j})t$

$= (t+1)\underset{\sim}{i} + (3t-5)\underset{\sim}{j}$

The particles meet when $\underset{\sim}{r}_A = \underset{\sim}{r}_B$:

You want to show that A and B are never in the same place at the same time. So try to find the value of t when the two displacements are equal and show that this is impossible.

$3t\underset{\sim}{i} - 2t\underset{\sim}{j} = (t+1)\underset{\sim}{i} + (3t-5)\underset{\sim}{j}$

$$\Rightarrow \begin{cases} 3t = t+1 \\ -2t = 3t-5 \end{cases}$$

If two vectors are equal then both components have to be equal. So you need a value of t that works in both equations.

From the first equation:

$2t = 1 \Rightarrow t = \frac{1}{2}$

Check in the second equation:

$-2\left(\frac{1}{2}\right) = -1$

$3\left(\frac{1}{2}\right) - 5 = -\frac{7}{2}$

Hence $\underset{\sim}{r}_A \neq \underset{\sim}{r}_B$ for all t so the particles never meet.

There is not a value of t which makes the two position vectors equal.

c At time t,

The distance between the particles is the magnitude of the displacement between them, which is found by subtracting the two position vectors.

$\overrightarrow{AB} = \underset{\sim}{r}_B - \underset{\sim}{r}_A$

$= (t+1)\underset{\sim}{i} + (3t-5)\underset{\sim}{j} - (3t\underset{\sim}{i} - 2t\underset{\sim}{j})$

$= (1-2t)\underset{\sim}{i} + (5t-5)\underset{\sim}{j}$

The distance between A and B is:

$AB = \sqrt{(1-2t)^2 + (5t-5)^2}$

$AB^2 = (1 - 4t + 4t^2) + (25t^2 - 50t + 25)$

$= 29t^2 - 54t + 26$

This expression has a minimum value when its square has a minimum value; so look at AB^2 to avoid having to work with the square root.

Let $y = 29t^2 - 54t + 26$

You could complete the square to find the minimum value, but it is not easy with these numbers so differentiate instead.

Then $\frac{dy}{dt} = 58t - 54 = 0$

$\Rightarrow t = \frac{27}{29}$

The minimum value of y is:

Don't forget that this is the minimum value for AB^2.

$29\left(\frac{27}{29}\right)^2 - 54\left(\frac{27}{29}\right) + 26 = 0.862$

Hence the minimum distance AB is

$\sqrt{0.862} = 0.928$ m

EXERCISE 16A

1 Points A, B and C have position vectors $4\mathbf{i}-3\mathbf{j}$, $\mathbf{i}+2\mathbf{j}$ and $-5\mathbf{i}+\mathbf{j}$, where distance is measured in metres. Find the average velocity if the particle travels:

a **i** from A to B in 3 seconds **ii** from A to C is 4 seconds

b **i** from C to B in 5 seconds **ii** from B to A in 4 seconds

c **i** from A to B in 3 seconds and then from B to C is 5 seconds

ii from C to A in 7 seconds and then from A to B in 4 seconds.

2 Find the average acceleration vector, and the magnitude of average acceleration in each case.

a **i** The velocity changes from $\begin{pmatrix}6\\-2\end{pmatrix}$ m s^{-1} to $\begin{pmatrix}8\\3\end{pmatrix}$ m s^{-1} in 10 seconds.

ii The velocity changes from $\begin{pmatrix}-3\\5\end{pmatrix}$ m s^{-1} to $\begin{pmatrix}1\\10\end{pmatrix}$ m s^{-1} in 8 seconds.

b **i** A particle accelerates from rest to $(4\mathbf{i}-2\mathbf{j})$ m s^{-1} in 5 seconds.

ii A particle accelerates from rest to $(-3\mathbf{i}+4\mathbf{j})$ m s^{-1} in 10 seconds.

3 Three points have coordinates $A(3, 5)$, $B(12, 7)$ and $C(8, 0)$.

a A particle travels in a straight line from A to B in 6 seconds.

Find its average velocity and average speed.

b Another particle travels in a straight line from B to C in 8 seconds and then in a straight line from C to B in 5 seconds.

Find its average velocity and average speed.

4 A particle moves in the plane so that its displacement from the origin at time t is given by the vector $\begin{pmatrix}t-3\\2+t^2\end{pmatrix}$.

a Find the particle's distance from the origin when $t=2$.

b Find the Cartesian equation of the particle's trajectory.

5 **a** An object's velocity changes from $(5\mathbf{i}-2\mathbf{j})$ m s^{-1} to $(3\mathbf{i}+4\mathbf{j})$ m s^{-1} in 3 seconds.

Find the magnitude of its average acceleration.

b The object then moves for another 10 seconds with average acceleration $(-\mathbf{i}+0.5\mathbf{j})$ m s^{-2}.

Find its direction of motion at the end of the 10 seconds.

6 A particle travels in a straight line from point P, with coordinates $(-4, 7)$, to point Q with coordinates $(3, -2)$. The journey takes 12 seconds and the distance is measured in metres.

a Find the average speed of the particle.

The particle then takes a further 7 seconds to travel in a straight line to point R with coordinates $(2, 5)$.

b Find the displacement from P to R.

c Find the average velocity of the particle for the whole journey.

d Find the average speed for the whole journey from P to R.
Explain why this is not equal to the magnitude of the average velocity.

7 Two particles, A and B, move in the plane. A has constant velocity $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ m s^{-1} and its initial displacement from the origin is $\begin{pmatrix} 14 \\ 0 \end{pmatrix}$ m. B starts from the origin and moves with constant velocity $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ m s^{-1}.
Show that the two particles meet and find the position vector of the meeting point.

8 A particle moves in a plane so that this displacement from the origin at time $t \geqslant 0$ is given by the vector $(t-1)\mathbf{i} + (6 + 4t - t^2)\mathbf{j}$ m.

a Find the distance of the particle from the origin when $t = 3$.

b Sketch the trajectory of the particle.

9 An object moves with a constant velocity $(-2\mathbf{i} + \mathbf{j})$ m s^{-1}. Its initial displacement from the origin is $(3\mathbf{i} - 4\mathbf{j})$ m.

a Find the Cartesian equation of the particle's trajectory.

b Find the minimum distance of the particle from the origin.

10 A particle moves in the plane so that its displacement from the origin at time t seconds is $(4\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j})$ m

Find the maximum distance of the particle from the origin.

Section 2: Constant acceleration equations

When a particle moves with constant acceleration, you can use formulae analogous to those for one-dimensional motion.

Key point 16.2

Constant acceleration formulae in two dimensions:

- $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
- $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
- $\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

Notice that for the formulae for position vector (the second and third formulae), an initial position vector $\mathbf{r}_0$ is needed in case the particle does not start at the origin. The second formula is just an extension of the first part of Key point 16.1, as the velocity is no longer constant here.

Rewind

See Student Book 1 for a reminder of the constant acceleration formulae.

Fast forward

The list in Key point 16.2 does not contain the vector version of the formula $v^2 = u^2 + 2as$. If you study Further Mathematics, you will meet a way of multiplying vectors (called the scalar product) that enables you to extend this formula to two dimensions as well.

WORKED EXAMPLE 16.5

A particle starts with initial velocity $(3\mathbf{i} - \mathbf{j})$ m s^{-1} and moves with constant acceleration. After 5 seconds its velocity is $(1.5\mathbf{i} + 2\mathbf{j})$ m s^{-1}. Find:

a the displacement from its initial position

b the distance from the initial position at this time.

a $\underline{u} = 3\underline{i} - \underline{j}$

$\underline{v} = 1.5\underline{i} + 2\underline{j}$

$t = 5$

$\underline{r} = ?$

Write down what you know and what you want. Notice that $\mathbf{r}_0$ is not known here but since you want the displacement from the initial position, it doesn't matter where it starts.

$$\underline{r} = \underline{r}_o + \frac{1}{2}(\underline{u} + \underline{v})t$$

Use $\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}(\mathbf{u} + \mathbf{v})t$.

$$= \underline{r}_o + \frac{1}{2}((3\underline{i} - \underline{j}) + (1.5\underline{i} + 2\underline{j})) \times 5$$

$$= \underline{r}_o + 2.5(4.5\underline{i} + \underline{j})$$

$$= \underline{r}_o + (11.25\underline{i} + 2.5\underline{j})$$

So, displacement from $\underline{r}_o$ is $(11.25\underline{i} + 2.5\underline{j})$ m

Displacement from $\mathbf{r}_0$ is $\mathbf{r} - \mathbf{r}_0$.

b Distance from starting position:

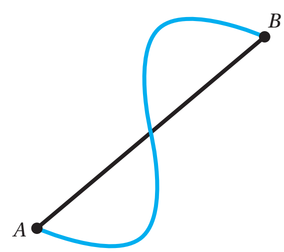
$\sqrt{11.25^2 + 2.5^2} = 11.5$ m

Distance is the magnitude of the displacement.

Tip

If you just want the displacement from the initial position and not the final displacement vector, then use displacement = $\mathbf{r} - \mathbf{r}_0$.

Since the particle does not move in a straight line, the distance from the starting point (which is measured in a straight line) is not the same as distance travelled (which is along a curve).



You need to be a little careful when solving equations with vectors. If you are comparing two sides of a vector equation, corresponding components need to be equal.

WORKED EXAMPLE 16.6

A particle moves with constant acceleration $\begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$ m s^{-2}. It is initially at the origin and its initial velocity is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ m s^{-1}.

Find the time when the particle is at the point with position vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ m.

$\underline{u} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

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$\underline{a} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\underline{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$t = ?$

Write down what you know and what you want.

$\underline{r} = \underline{r}_0 + \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}t + \frac{1}{2}\begin{pmatrix} -1.5 \\ 3 \end{pmatrix}t^2$

Use $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

First component:

$1 = 2t - 0.75t^2$

$0.75t^2 - 2t + 1 = 0$

$t = 2$ or $\frac{2}{3}$

The first component of the vectors gives a quadratic equation for t. This will give two possible values.

Second component:

when $t = 2$: $5t + \frac{1}{2}(3)t^2 = 16$

when $t = \frac{2}{3}$: $5t + \frac{1}{2}3t^2 = 4$

You need to check for which of these values of t the second component equals 4.

The particle is at $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ m when $t = \frac{2}{3}$ seconds.

WORK IT OUT 16.1

A particle moves with constant acceleration, starting from the origin. Its position vector at time t is given by $\mathbf{r} = (t^2 - 3t)\mathbf{i} + (2t^2 - 15t)\mathbf{j}$.

How many times does the particle pass through the origin during the subsequent motion?

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
At the origin the displacement is zero. When $t^2 - 3t = 0$ $t = 0$ or 3 So the particle passes through the origin once, when $t = 3$.	At the origin both components of the displacement are zero. When $t^2 - 3t = 0$ $t = 0$ or 3 When $2t^2 - 15t = 0$ $t = 0$ or 7.5 So the particle passes through the origin twice, when $t = 3$ and 7.5	At the origin both components of the displacement are zero. When $t^2 - 3t = 0$ $t = 0$ or 3 When $2t^2 - 15t = 0$ $t = 0$ or 7.5 So the particle does not pass through the origin again.

You saw in Student Book 1 that Newton's second law still applies in two dimensions: $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and m is a scalar. This means that the particle accelerates in the direction of the net force. You can now use this in conjunction with the constant acceleration formulae for vectors.

WORKED EXAMPLE 16.7

A particle of mass 2.4 kg moves under the action of a constant force, $\mathbf{F}$ N. When $t = 0$ the particle is at the origin moving with velocity $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ m s^{-1} and when $t = 4$ seconds its position vector is $\begin{pmatrix} 16 \\ -4 \end{pmatrix}$ m. Find the vector $\mathbf{F}$.

$\underline{u} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 16 \\ -4 \end{pmatrix}$

$\underline{r}_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$t = 4$

$\underline{a} = ?$

Find the acceleration and then use $\mathbf{F} = m\mathbf{a}$.

Since the force is constant the acceleration is also constant, so you can use the constant acceleration equations.

$$\underline{r} = \underline{r}_o + \underline{u}t + \frac{1}{2}\underline{a}t^2$$

$$\begin{pmatrix} 16 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 5 \end{pmatrix} + \frac{1}{2}(4^2)\underline{a}$$

$$8\underline{a} = \begin{pmatrix} 16 \\ -4 \end{pmatrix} - \begin{pmatrix} -8 \\ 20 \end{pmatrix}$$

Rearrange to find $\mathbf{a}$

$$= \begin{pmatrix} 24 \\ 16 \end{pmatrix}$$

$$\underline{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ m s}^{-2}$$

$$\underline{F} = m\underline{a}$$

Now use $\mathbf{F} = m\mathbf{a}$.

$$= 2.4\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7.2 \\ 4.8 \end{pmatrix} \text{ N}$$

EXERCISE 16B

1 In each question the particle moves with constant acceleration. It is initially at the origin. Time is measured in seconds and displacement in metres.

a **i** $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} -0.6 \\ 0.7 \end{pmatrix}$, find $\mathbf{v}$ when $t = 4$ **ii** $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{a} = 1.2\mathbf{i} - 0.6\mathbf{j}$, find $\mathbf{v}$ when $t = 7$

b **i** $\mathbf{u} = -2\mathbf{i} + 0.5\mathbf{j}$, $\mathbf{a} = 0.3\mathbf{i} - 0.8\mathbf{j}$, find $\mathbf{r}$ when $t = 5$ **ii** $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} -0.6 \\ 0.7 \end{pmatrix}$, find $\mathbf{r}$ when $t = 3$

c **i** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{s} = -2\mathbf{i} + \mathbf{j}$, $t = 4$, find $\mathbf{u}$. **ii** $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $t = 6$, find $\mathbf{u}$.

d **i** $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$, find t. **ii** $\mathbf{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 48 \\ 18 \end{pmatrix}$, find t.

2 An object of mass m kg moves under the action of the force $\mathbf{F}$ N. The object is initially at rest. Find the speed of the object at time t seconds, in each case.

a **i** $m = 6, t = 5, \mathbf{F} = 24\mathbf{i} + 6\mathbf{j}$ **ii** $m = 2, t = 10, \mathbf{F} = 6\mathbf{i} + 10\mathbf{j}$

b **i** $m = 0.2, t = 7, \mathbf{F} = 6\mathbf{i} - 2\mathbf{j}$ **ii** $m = 0.5, t = 5, \mathbf{F} = -3\mathbf{i} + 9\mathbf{j}$

3 An object moves with constant acceleration $\begin{pmatrix} 0.6 \\ -0.4 \end{pmatrix}$ m s^{-2} and initial velocity $\begin{pmatrix} 3.5 \\ 2.4 \end{pmatrix}$ m s^{-1}.

Find its velocity and the displacement from the initial position after 7 seconds.

4 A particle moves with constant acceleration $(3\mathbf{i} - \mathbf{j})$ m s^{-2}. It is initially at the origin and its velocity is $(2\mathbf{i} + 5\mathbf{j})$ m s^{-1}.

a Find the distance of the particle from the origin after 3 seconds.

b Find the direction of motion of the particle at this time.

5 A particle passes the origin with velocity $(2\mathbf{i} + 5\mathbf{j})$ m s^{-1} and moves with constant acceleration.

a Given that 7 seconds later its velocity is $(-12\mathbf{i} + 15.5\mathbf{j})$ m s^{-1}, find the acceleration.

b Find the time when the particle's displacement from the origin is $(-8\mathbf{i} + 32\mathbf{j})$ m.

6 An object moves with constant acceleration. When $t = 0$ s it has velocity $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ m s^{-1}.
When $t = 5$ s its displacement from the initial position is $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ m.

Find the magnitude of the acceleration.

7 A particle moves with constant acceleration. Its initial velocity is $(3\mathbf{i} - 2\mathbf{j})$ m s^{-1}.
8 seconds later, its displacement from the initial position is $(-44\mathbf{i} + 20\mathbf{j})$ m.
Find its direction of motion at this time.

8 An object moves with constant acceleration and initial velocity $5\mathbf{j}$ m s^{-1}.
When its displacement from the initial position is $(12.5\mathbf{i} + 5\mathbf{j})$ m, its velocity is $(5\mathbf{i} - 3\mathbf{j})$ m s^{-1}.

Find the magnitude of the acceleration.

9 A particle moves with constant acceleration $\begin{pmatrix} 3.8 \\ 2.2 \end{pmatrix}$ m s^{-2}. Given that its initial velocity is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ m s^{-1}, find the time when its displacement from the initial position is $\begin{pmatrix} 180 \\ 130 \end{pmatrix}$ m.

10 A particle of mass 2.5 kg is subjected to a constant force $\mathbf{F} = (1.2\mathbf{i} + 0.9\mathbf{j})$ N. The initial velocity of the particle is $(0.6\mathbf{i} - 1.3\mathbf{j})$ m s^{-1}.

Find the velocity of the particle after 5 seconds.

11 A particle starts with initial velocity $6\mathbf{j}$ m s^{-1} and moves with constant acceleration $0.5\mathbf{i}$ m s^{-2}.

Prove that the speed of the particle increases with time.

12 A particle moves with constant acceleration $\mathbf{a} = (-2\mathbf{i} + \mathbf{j})$ m s^{-2}. When $t = 0$ s the particle is at rest, at the point with the position vector $(5\mathbf{i} + 3\mathbf{j})$ m.

Find the shortest distance of the particle from the origin during the subsequent motion.

Section 3: Calculus with vectors

When the acceleration is not constant you need to use differentiation and integration to find expressions for displacement and velocity. In Student Book 1, you learnt how to do that for motion in one dimension. The same principles apply to two-dimensional motion: differentiating the displacement equation gives the velocity equation, and differentiating the velocity equation gives the acceleration equation. The only difference is that those quantities are now represented by vectors.

Key point 16.3

To differentiate or integrate a vector, differentiate or integrate each component separately.

WORKED EXAMPLE 16.8

A particle moves in two dimensions. Its position vector, measured in metres, varies with time (measured in seconds) as $\mathbf{r} = \begin{pmatrix} 2t^2 - 4 \\ 1 - t^3 \end{pmatrix}$.

Find the speed of the particle when $t = 3$ s.

$$\underline{v} = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \begin{pmatrix} 4t \\ -3t^2 \end{pmatrix}$$

To find the velocity, differentiate the displacement vector. Do this by differentiating each component separately.

When $t = 3$:

$$\underline{v} = \begin{pmatrix} 12 \\ -27 \end{pmatrix}$$

$$\text{Speed} = |\underline{v}| = \sqrt{12^2 + 27^2} = 29.5 \text{ m s}^{-1}$$

Speed is the magnitude of velocity.

When using integration with vectors, the constant of integration will also be a vector.

WORKED EXAMPLE 16.9

A particle moves with acceleration $\mathbf{a} = \begin{pmatrix} 2t+1 \\ 2\sin t \end{pmatrix}$ m s^{-2}. The initial velocity is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ m s^{-1}.

Find the expression for the velocity at time t.

$$\underline{v} = \int \underline{a}\, dt$$

To find the velocity, integrate the acceleration vector. Do this by integrating each component separately.

$$= \int \begin{pmatrix} 2t+1 \\ 2\sin t \end{pmatrix} dt$$

$$= \begin{pmatrix} t^2+t \\ -2\cos t \end{pmatrix} + \mathbf{c}$$

When $t = 0$, $\underline{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$:

Use the initial velocity to find c.

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \underline{c}$$

$$\Rightarrow \underline{c} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

So $\underline{v} = \begin{pmatrix} t^2+t-1 \\ -2\cos t+5 \end{pmatrix}$ m s^{-1}

You can include the constant within the existing vector.

Remember that, for two vectors to be equal, corresponding components need to be equal.

WORKED EXAMPLE 16.10

A particle starts from point P with the velocity $(3\mathbf{i} + \mathbf{j})$ m s^{-1}. Its acceleration is given by $\mathbf{a} = (-t\,\mathbf{i} + 2t\,\mathbf{j})$ m s^{-2}.

Show that the particle never returns to P.

Let $\underline{r}_0$ be the position vector of P.

You need to find an expression for the position vector, $\mathbf{r}$, and show that it never equals $\mathbf{r}_0$ for $t > 0$.

$$\underline{v} = \int (-t\underline{i} + 2t\underline{j})\, dt$$

First integrate $\mathbf{a}$ to find $\mathbf{v}$.

$$= -\frac{1}{2}t^2\underline{i} + t^2\underline{j} + \underline{c}$$

Continues on next page

$\underset{\sim}{v} = 3\underset{\sim}{i} + \underset{\sim}{j}$ when $t = 0$, so $\underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j}$.

Use the initial velocity to find **c**.

$\therefore \underset{\sim}{v} = \left(-\frac{1}{2}t^2 + 3\right)\underset{\sim}{i} + (t^2 + 1)\underset{\sim}{j}$

$\underset{\sim}{r} = \int \left[\left(-\frac{1}{2}t^2 + 3\right)\underset{\sim}{i} + (t^2 + 1)\underset{\sim}{j}\right] \mathrm{d}t$

$= \left(-\frac{1}{6}t^3 + 3t\right)\underset{\sim}{i} + \left(\frac{1}{3}t^3 + t\right)\underset{\sim}{j} + \underset{\sim}{r}_0$

Now integrate the velocity to find the position vector. Initially $\mathbf{r} = \mathbf{r}_0$, so $\mathbf{c} = \mathbf{r}_0$.

When $\underset{\sim}{r} = \underset{\sim}{r}_0$:

$-\frac{1}{6}t^3 + 3t = 0$

$\frac{1}{6}t(-t^2 + 18) = 0$

$t = 0$ or $\sqrt{18}$

Now check if there is any value of t (other than $t = 0$) when $\mathbf{r} = \mathbf{r}_0$. Check both components of the vector.

When $t = \sqrt{18}$:

$\frac{1}{3}t^3 + t = 19.7 \neq 0$

$t = 0$ is the starting point so use $t = \sqrt{18}$.

Hence $\underset{\sim}{r} \neq \underset{\sim}{r}_0$ for $t > 0$, so the particle does not return to the starting point.

WORK IT OUT 16.2

A particle starts from rest and moves with acceleration $\mathbf{a} = \begin{pmatrix} 3\sin t \\ -5\cos t \end{pmatrix}$ m s^{-2}.
Find an expression for the velocity of the particle.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{v} = \int \begin{pmatrix} 3\sin t \\ -5\cos t \end{pmatrix} \mathrm{d}t$ $= \begin{pmatrix} -3\cos t \\ -5\sin t \end{pmatrix} + c$ Initially at rest means that the speed is zero. When $t = 0$: $\mathbf{v} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ $\Rightarrow$ speed $= \sqrt{(-3)^2 + 0^2} = 3$ $\Rightarrow c = -3$ $\mathbf{v} = \begin{pmatrix} -3\cos t - 3 \\ -5\sin t - 3 \end{pmatrix}$ m s^{-1}	$\mathbf{v} = \int \begin{pmatrix} 3\sin t \\ -5\cos t \end{pmatrix} \mathrm{d}t$ $= \begin{pmatrix} -3\cos t \\ -5\sin t \end{pmatrix} = \mathbf{c}$ Initially at rest: $\mathbf{v}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \mathbf{c}$ $\Rightarrow \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\therefore \mathbf{v} = \begin{pmatrix} -3\cos t + 3 \\ -5\sin t \end{pmatrix}$ m s^{-1}	$\mathbf{v} = \int \begin{pmatrix} 3\sin t \\ -5\cos t \end{pmatrix} \mathrm{d}t$ $= \begin{pmatrix} -3\cos t \\ -5\sin t \end{pmatrix} = \mathbf{c}$ Initially at rest, so $\mathbf{c} = \mathbf{0}$ $\therefore \mathbf{v} = \begin{pmatrix} -3\cos t \\ -5\sin t \end{pmatrix}$ m s^{-1}

When the force is variable (a function of t) you can still use $\mathbf{F} = m\mathbf{a}$, but because the acceleration will now be variable, you need to use calculus rather than the constant acceleration formulae.

WORKED EXAMPLE 16.11

A particle of mass 0.5 kg starts from rest and moves under the action the force $((4t)\mathbf{i} + (2t-2)\mathbf{j})$ N.

Find:

a the speed **b** the direction of motion of the particle after 3 seconds.

a $\underline{F} = m\underline{a}$

$(4t)\underline{i} + (2t-2)\underline{j} = 0.5\underline{a}$

$\Rightarrow \underline{a} = (8t)\underline{i} + (4t-4)\underline{j}\ \text{m s}^{-2}$

Find the velocity by integrating the acceleration. Use $\mathbf{F} = m\mathbf{a}$ first.

$\underline{v} = \int (8t)\underline{i} + (4t-4)\underline{j}\ \mathrm{d}t$

$= (4t^2)\underline{i} + (2t^2 - 4t)\underline{j} + \underline{c}$

$\underline{v}(0) = 0 \Rightarrow \underline{c} = \underline{0}$

Remember to find **c**.

$\therefore \underline{v} = ((4t^2)\underline{i} + (2t^2 - 4t)\underline{j})\ \text{m s}^{-1}$

When $t = 3$ s:

You can now use the given value of t to find the velocity vector.

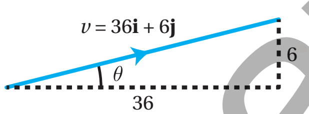
$\underline{v} = (36\underline{i} + 6\underline{j})\ \text{m s}^{-1}$

The speed is:

Speed is the magnitude of the velocity vector.

$|\underline{v}| = \sqrt{36^2 + 6^2} = 36.5\ \text{m s}^{-1}$

b



The direction is given by the angle with the vector **i**.

It is always a good idea to draw a diagram to see which angle you are looking for.

$\tan\theta = \frac{6}{36}$

$\theta = \tan^{-1}\left(\frac{6}{36}\right) = 9.46°$

The direction of motion is 9.46° above the horizontal.

EXERCISE 16C

1 For the particle moving with the given displacement, find expressions for the velocity and acceleration vectors. Also find the speed when $t = 3$ s.

a **i** $\mathbf{r} = (3t - \sin t)\mathbf{i} + (t - t^2)\mathbf{j}$ m **ii** $\mathbf{r} = (e^{2t} - t)\mathbf{i} + (t^2 + e^{2t})\mathbf{j}$ m

b **i** $\mathbf{r} = \begin{pmatrix} 4\cos 3t \\ 3\sin 2t \end{pmatrix}$ m **ii** $\mathbf{r} = \begin{pmatrix} 3\ln(t+1) - t \\ t^2 + \ln(t+1) \end{pmatrix}$ m

2 For a particle moving with the given acceleration, find expressions for the velocity and displacement vectors.

The initial displacement is zero, and the initial velocity is given in each question. Also find the distance from the starting point when $t = 3$.

a **i** $\mathbf{a} = (3 - t^2)\mathbf{i} + 2t\,\mathbf{j}$ m s^{-2}, $\mathbf{v}(0) = 2\mathbf{i} + 5\mathbf{j}$ m s^{-1} **ii** $\mathbf{a} = (t+1)\mathbf{i} + 3\mathbf{j}$ m s^{-2}, $\mathbf{v}(0) = \mathbf{i} - 2\mathbf{j}$ m s^{-1}

b **i** $\mathbf{a} = \begin{pmatrix} 3\,\mathrm{e}^{t} \\ 2\,\mathrm{e}^{-t} \end{pmatrix}$ m s^{-2}, $\mathbf{v}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ m s^{-1} **ii** $\mathbf{a} = \begin{pmatrix} 3\sin 2t \\ 3\cos 2t \end{pmatrix}$ m s^{-2}, $\mathbf{v}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ m s^{-1}

c **i** $\mathbf{a} = 2\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$ m s^{-2}, $\mathbf{v}(0) = \mathbf{0}$ m s^{-1} **ii** $\mathbf{a} = \mathrm{e}^{2t}\mathbf{i} + 3\mathrm{e}^{t}\mathbf{j}$ m s^{-2}, $\mathbf{v}(0) = \mathbf{0}$ m s^{-1}

3 A particle moves in a plane with the displacement from the starting point given by $\mathbf{r} = \mathrm{e}^{2t}\mathbf{i} + (t-1)\mathbf{j}$.

a Find an expression for the velocity of the particle at time t.

b Find the speed of the particle when $t = 5$.

4 A particle moves in a plane, starting from rest. Its acceleration varies according to the equation

$\mathbf{a} = \begin{pmatrix} 6t \\ \cos 2t \end{pmatrix}$ m s^{-2}.

a Find an expression for the velocity of the particle at time t.

b Find the displacement from the initial position after 3 seconds.

5 The velocity of a particle, in m s^{-1}, moving in a plane is given by $\mathbf{v} = (3 - \sin(2t))\mathbf{i} + 2\cos(2t)\mathbf{j}$.

a Find the initial speed of the particle.

b Find the magnitude of the acceleration when $t = 12$.

c Find an expression for the displacement from the initial position after t seconds.

6 A particle starts from rest and moves with acceleration $((2 + \mathrm{e}^{-2t})\mathbf{i} + 4\mathrm{e}^{-2t}\mathbf{j})$ m s^{-2}.
Find its distance from the initial position after 1.2 seconds.

7 The velocity of a particle moving in a plane is given by $\mathbf{v} = \begin{pmatrix} 2 - 3t^2 \\ 4t - 1 \end{pmatrix}$ m s^{-1}.

Show that the particle never returns to its initial position.

8 For a particle moving in two dimensions, the displacement vector from the starting point is given by

$\mathbf{r} = \begin{pmatrix} 3t^3 - 4t \\ t^4 - 2t^3 + t \end{pmatrix}$

a The components of the displacement vector give parametric equations of the trajectory of the particle, $x = x(t)$, $y = y(t)$. Use parametric differentiation to find the gradient of the tangent to this curve, $\frac{\mathrm{d}y}{\mathrm{d}x}$, when $t = 3$.

b Find the velocity vector when $t = 3$. What do you notice?

9 A particle of mass 2 kg moves under the action of the force $\mathbf{F} = (24\cos(2t)\mathbf{i} - 24\sin(2t)\mathbf{j})$ N.
Its initial velocity is $\mathbf{v}(0) = (6\mathbf{j})$ m s^{-1}.

a Show that the speed of the particle is constant.

b By considering the x and y components of the displacement vector, show that the particle moves in a circle.

10 A particle moves in the plane, from the initial position $\begin{pmatrix}5\\0\end{pmatrix}$ m. Its velocity, $\mathbf{v}$ m s^{-1}, at time t s is given by the equation: $\mathbf{v} = \begin{pmatrix}-8t\\2\end{pmatrix}$.

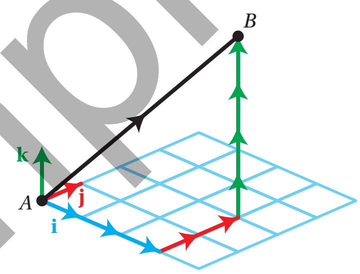
Find the time when the particle is closest to the origin, and find this minimum distance.

Section 4: Vectors in three dimensions

In the preceding sections you learnt how to use vector equations to represent motion in two dimensions. Now vector methods will be extended to enable you to describe positions and various types of motion in the three-dimensional world.

To represent positions and displacements in three-dimensional space, you need three base vectors, all perpendicular to each other. They are conventionally called **i**, **j**, **k**.

You can also show the components in a column vector: $\overrightarrow{AB} = \begin{pmatrix}3\\2\\4\end{pmatrix}$.



$\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

Each point in a three-dimensional space can be represented by a position vector, which equals its displacement from the origin. The displacement from one point to another is the difference between their position vectors.

WORKED EXAMPLE 16.12

Points A and B have coordinates $(3, -1, 2)$ and $(5, 0, 3)$, respectively. Write as column vectors:

a the position vectors of A and B

b the displacement vector $\overrightarrow{AB}$.

a $\mathbf{a} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}5\\0\\3\end{pmatrix}$

The components of the position vectors are the coordinates of the point.

b $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$= \begin{pmatrix}5\\0\\3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix}$

$= \begin{pmatrix}2\\1\\1\end{pmatrix}$

Relate $\overrightarrow{AB}$ to the position vectors **a** and **b** exactly as you would in two dimensions.

Did you know?

Although our space is three dimensional, it turns out that many situations can be modelled as motion in two dimensions. For example, it is possible to prove that the orbit of a planet lies in a plane, so two-dimensional vectors are sufficient to describe it.

The formula for the magnitude of a three-dimensional vector is analogous to the two-dimensional one.

Key point 16.4

- The magnitude (modulus) of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- The distance between points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{b} - \mathbf{a}|$.

WORKED EXAMPLE 16.13

Points A and B have position vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find the exact distance AB.

$$\begin{aligned}\overrightarrow{AB} &= \underline{b} - \underline{a} \\ &= (5\underline{i} + 2\underline{j} + 3\underline{k}) - (2\underline{i} - \underline{j} + 5\underline{k}) \\ &= 3\underline{i} + 3\underline{j} - 2\underline{k}\end{aligned}$$

The distance is the magnitude of the displacement vector, so you need to find $\overrightarrow{AB}$ first.

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{3^2 + 3^2 + 2^2} \\ &= \sqrt{22}\end{aligned}$$

Now use the formula for the magnitude.

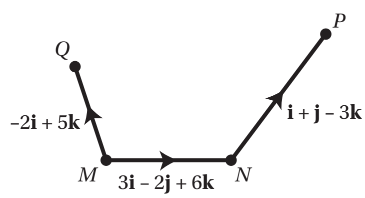
Remember that you can use vector addition and subtraction to combine displacements.

WORKED EXAMPLE 16.14

The diagram shows points M, N, P, Q such that $\overrightarrow{MN} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, $\overrightarrow{NP} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{MQ} = -2\mathbf{j} + 5\mathbf{k}$.

Write each vector in component form.

a $\overrightarrow{MP}$ **b** $\overrightarrow{PM}$ **c** $\overrightarrow{PQ}$.



a $$\begin{aligned}\overrightarrow{MP} &= \overrightarrow{MN} + \overrightarrow{NP} \\ &= (3\underline{i} - 2\underline{j} + 6\underline{k}) + (\underline{i} + \underline{j} - 3\underline{k}) \\ &= 4\underline{i} - \underline{j} + 3\underline{k}\end{aligned}$$

You can get from M to P via N.

b $$\begin{aligned}\overrightarrow{PM} &= -\overrightarrow{MP} \\ &= -4\underline{i} + \underline{j} - 3\underline{k}\end{aligned}$$

Going from P to M is the reverse of going from M to P.

c $$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PM} + \overrightarrow{MQ} \\ &= (-4\underline{i} + \underline{j} - 3\underline{k}) + (-2\underline{j} + 5\underline{k}) \\ &= -4\underline{i} - \underline{j} + 2\underline{k}\end{aligned}$$

You can get from P to Q via M, using the answers from previous parts.

EXERCISE 16D

1 Write each vector in column vector notation (in three dimensions).

a i $4\mathbf{i}$ ii $-5\mathbf{j}$

b i $3\mathbf{i}+\mathbf{k}$ ii $2\mathbf{j}-\mathbf{k}$

2 Let $\mathbf{a}=\begin{pmatrix}7\\1\\12\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}5\\-2\\3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}1\\1\\2\end{pmatrix}$. Find each vector.

a i $3\mathbf{a}$ ii $4\mathbf{b}$ b i $\mathbf{a}-\mathbf{b}$ ii $\mathbf{b}+\mathbf{c}$

c i $2\mathbf{b}+\mathbf{c}$ ii $\mathbf{a}-2\mathbf{b}$ d i $\mathbf{a}+\mathbf{b}-2\mathbf{c}$ ii $3\mathbf{a}-\mathbf{b}+\mathbf{c}$

3 Let $\mathbf{a}=\mathbf{i}+2\mathbf{j}$, $\mathbf{b}=\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$. Find each vector.

a i $-5\mathbf{b}$ ii $4\mathbf{a}$

b i $\mathbf{c}-\mathbf{a}$ ii $\mathbf{a}-\mathbf{b}$

c i $\mathbf{a}-\mathbf{b}+2\mathbf{c}$ ii $4\mathbf{c}-3\mathbf{b}$

4 Find the magnitude of each vector in three dimensions.

$\mathbf{a}=\begin{pmatrix}4\\1\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\mathbf{c}=2\mathbf{i}-4\mathbf{j}+\mathbf{k}$, $\mathbf{d}=\mathbf{j}-\mathbf{k}$

5 Find the distance between each pair of points in three dimensions.

a i $A(1,0,2)$ and $B(2,3,5)$ ii $C(2,1,7)$ and $D(1,2,1)$

b i $P(3,-1,-5)$ and $Q(-1,-4,2)$ ii $M(0,0,2)$ and $N(0,-3,0)$

6 Find the distance between the points with the given position vectors.

a $\mathbf{a}=2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}-6\mathbf{k}$

b $\mathbf{a}=\begin{pmatrix}3\\7\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\-2\\-5\end{pmatrix}$

c $\mathbf{a}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}0\\0\\5\end{pmatrix}$

d $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and $\mathbf{b}=\mathbf{j}-\mathbf{k}$

7 Given that $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$, find the vector $\mathbf{b}$ such that:

a $\mathbf{a}+\mathbf{b}$ is the zero vector b $2\mathbf{a}+3\mathbf{b}$ is the zero vector

c $\mathbf{a}-\mathbf{b}=\mathbf{j}$ d $\mathbf{a}+2\mathbf{b}=3\mathbf{i}$.

8 Given that $\mathbf{a}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}5\\3\\3\end{pmatrix}$ find vector $\mathbf{x}$ such that $3\mathbf{a}+4\mathbf{x}=\mathbf{b}$.

9 Given that $\mathbf{a}=3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{c}=\mathbf{i}+\mathbf{k}$, find the value of the scalar t such that $\mathbf{a}+t\mathbf{b}=\mathbf{c}$.

10 Find the possible values of the constant c such that the vector $\begin{pmatrix}2c\\c\\-c\end{pmatrix}$ has magnitude 12.

11 Let $\mathbf{a}=\begin{pmatrix}-2\\0\\-1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a}+\lambda\mathbf{b}|=5\sqrt{2}$.

Elevate

See Support sheet 16 for a further example of three-dimensional vectors and for more practice questions.

12 Points A and B are such that $\overrightarrow{OA}=\begin{pmatrix}-1\\-6\\13\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}1\\-2\\4\end{pmatrix}+t\begin{pmatrix}2\\1\\-5\end{pmatrix}$

where O is the origin. Find the possible values of t such that $AB=3$.

13 Points P and Q have position vectors $\mathbf{p}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $\mathbf{q}=(2+t)\mathbf{i}+(1-t)\mathbf{j}+(1+t)\mathbf{k}$. Find the value of t for which the distance PQ is minimum possible and find this minimum distance.

Section 5: Solving geometrical problems

This chapter finishes with a review of how you can use vector methods to solve geometrical problems. You have already used these results:

- the position vector of the midpoint of line segment AB is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
- if vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel then there is a scalar k so that $\mathbf{b}=k\mathbf{a}$
- the unit vector in the same direction as $\mathbf{a}$ is $\hat{\mathbf{a}}=\frac{1}{|\mathbf{a}|}\mathbf{a}$.

Rewind

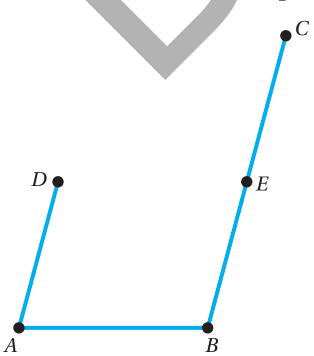
These results were introduced in Student Book 1.

WORKED EXAMPLE 16.15

Points A, B, C, and D have position vectors $\mathbf{a}=\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}5\\0\\3\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}7\\8\\-3\end{pmatrix}$, $\mathbf{d}=\begin{pmatrix}4\\3\\-2\end{pmatrix}$.

Point E is the midpoint of BC.

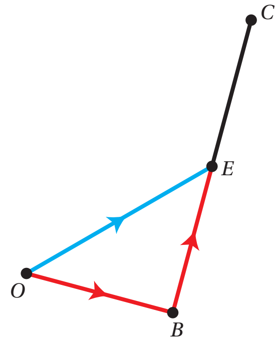
a Find the position vector of E.
b Show that $ABED$ is a parallelogram.



Draw a diagram to show what is going on.

Continues on next page

a



As you are given position vectors, it may help to show the origin on the diagram. For this part, you only need to look at points B, C and E.

$$\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE}$$
$$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$$
$$= \underline{b} + \frac{1}{2}(\underline{c} - \underline{b})$$
$$= \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$$
$$= \begin{pmatrix} 2.5 \\ 0 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 4 \\ -1.5 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

Tip

Vector diagrams do not have to be accurate or to scale to be useful. A two-dimensional sketch of a three-dimensional situation is often enough to show you what's going on.

b $\overrightarrow{AD} = \underline{d} - \underline{a}$

$$= \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{BE} = \underline{e} - \underline{b}$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{AD} = \overrightarrow{BE}$

$\therefore$ $ABED$ is a parallelogram.

In a parallelogram, opposite sides are equal and parallel, which means that the vectors corresponding to those sides are equal. So you need to show that $\overrightarrow{AD} = \overrightarrow{BE}$.

WORKED EXAMPLE 16.16

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$

a Find the values of p and q such that $\mathbf{c}$ is parallel to $\mathbf{a}$.

b Find the value of scalar k such that $\mathbf{a} + k\mathbf{b}$ is parallel to vector $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

a Write $\underline{c} = t\underline{a}$ for some scalar t.

Then:

$$\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$$

If two vectors are parallel you can write $\mathbf{v}_1 = t\,\mathbf{v}_2$.

$$\begin{cases} -2 = t \\ p = 2 \\ q = 7 \end{cases}$$

If two vectors are equal then all their corresponding components are equal.

$\therefore p = -4, q = -14$

b $\underline{a} + k\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -3k \\ 4k \\ 2k \end{pmatrix} = \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix}$

Write vector $\mathbf{a} + k\mathbf{b}$ in terms of k and then use $\mathbf{a} + k\mathbf{b} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

Parallel to $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$:

$$\begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$$

$$\begin{cases} 1-3k = 0 \\ 2+4k = 10t \\ 7+2k = 23t \end{cases}$$

$1-3k = 0 \Rightarrow k = \frac{1}{3}$

$2+4\left(\frac{1}{3}\right) = 10t \Rightarrow t = \frac{1}{3}$

You can find k from the first equation, but you need to check that all three equations are satisfied.

Check in the third equation:

$\text{LHS} = 7+2\left(\frac{1}{3}\right) = \frac{23}{3}$

$\text{RHS} = 23\left(\frac{1}{3}\right) = \frac{23}{2}$

$\therefore k = \frac{1}{3}$

WORKED EXAMPLE 16.17

a Find the unit vector in the same direction as $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a vector of magnitude 5 parallel to $\mathbf{a}$.

a Let the required unit vector be $\hat{\underline{a}}$.

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

Find the magnitude of $\mathbf{a}$.

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

Then divide $\mathbf{a}$ by its magnitude to produce a vector in the same direction as $\mathbf{a}$ but of length 1.

$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \div 3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

b Let $\underline{b}$ be parallel to $\underline{a}$ and $|\underline{b}| = 5$. Then

To get a vector of magnitude 5, multiply the unit vector by 5.

$$\underline{b} = 5\hat{\underline{a}} = \begin{pmatrix} \frac{10}{3} \\ -\frac{10}{3} \\ \frac{5}{3} \end{pmatrix}$$

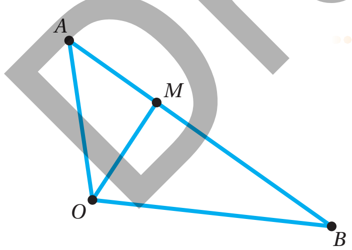
Tip

Note that part **b** has two possible answers, as **b** could be in the opposite direction. To get the second answer you would take the scalar to be –5 instead of 5.

The midpoint is just a special case of dividing a line segment in a given ratio.

WORKED EXAMPLE 16.18

Points A and B have position vectors **a** and **b**. Find, in terms of **a** and **b**, the position vector of the point M on AB such that $AM : MB = 2 : 3$.



Always start by drawing a diagram. Since the question is about position vectors, include the origin.

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

You can get from O to M via either A or B.

$$= \overrightarrow{OA} + \frac{2}{5}\overrightarrow{AB}$$

$AM : MB = 2 : 3$ means that $\overrightarrow{AM} = \frac{2}{5}\overrightarrow{AB}$

$$= \underline{a} + \frac{2}{5}(\underline{b} - \underline{a})$$

$$= \frac{3}{5}\underline{a} + \frac{2}{5}\underline{b}$$

EXERCISE 16E

1 **a** **i** Find a unit vector parallel to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

ii Find a unit vector parallel to $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$.

b **i** Find a unit vector in the same direction as $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

ii Find a unit vector in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

2 Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$.

a Write $\overrightarrow{AB}$ as a column vector.

b Find the position vector of the midpoint of AB.

3 Points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find the position vector of point D such that $ABCD$ is a parallelogram.

4 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ find the value of the scalar p such that $\mathbf{a} + p\mathbf{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

5 Given that $\mathbf{x} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{y} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ find the value of the scalar λ such that $\lambda\mathbf{x} + \mathbf{y}$ is parallel to vector $\mathbf{j}$.

6 Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. Point C lies on the line segment AB so that $AC : BC = 2 : 3$. Find the position vector of C.

7 Points P and Q have position vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

a Find the position vector of the midpoint M of PQ.

b Point R lies on the line PQ such that $QR = QM$. Find the coordinates of R.

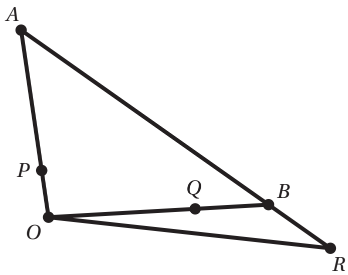
8 Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2q\mathbf{i} + \mathbf{j} + q\mathbf{k}$ find the values of scalars p and q such that $p\mathbf{a} + \mathbf{b}$ is parallel to vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

9 **a** Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

b Find a vector of magnitude 3 in the same direction as $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

10 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. Point M lies on AB and $AM : MB = p : q$. Express the position vector of M in terms of $\mathbf{a}$, $\mathbf{b}$, p and q.

11 In the diagram, O is the origin and points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. P, Q and R are points on OA, OB and AB such that $OP : PA = 1 : 4$, $OQ : QB = 3 : 2$ and $AB : BR = 5 : 1$.



Prove that:

a PQR is a straight line

b Q is the mid-point of PR.

Checklist of learning and understanding

- Constant acceleration formulae in two dimensions:
  - $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
  - $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
  - $\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}(\mathbf{u} + \mathbf{v})t$
- To differentiate or integrate a vector, differentiate or integrate each component separately.
- Vectors in three dimensions can be expressed in terms of **base vectors** $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ using **components**.
- The **magnitude** of a vector can be calculated using the components of the vector $|\mathbf{a}| = \sqrt{a_1 + a_2 + a_3}$.
- The **distance** between the points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $|\mathbf{b} - \mathbf{a}|$.
- The unit vector in the same direction as $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.

Mixed practice 16

1 The point A has position vector $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ and the point B has position vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$. Find the vector $\overrightarrow{AB}$.

Choose from these options.

A $\begin{pmatrix} 1 \\ -1 \\ -6 \end{pmatrix}$ **B** $\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$ **C** $\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ **D** $\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

2 A particle of mass 6 kg moves with constant acceleration $(1.6\mathbf{i} - 0.3\mathbf{j})$ m s^{-2}.

a Find the magnitude of the net force acting on the particle.

When $t = 0$ the particle has velocity $(-2\mathbf{i} + 2.5\mathbf{j})$ m s^{-1}.

b Find the speed and the direction of motion of the particle 5 seconds later.

3 Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. C is the midpoint of AB.

Find the exact distance AC.

4 Points A, B and C have position vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

a Find the position vector of the point D such that $ABCD$ is a parallelogram.

b Prove that $ABCD$ is a rhombus.

5 A particle moves in the plane so that its position vector at time t seconds is $(3.2 - t^2)\mathbf{i} + (-4.6 + 0.2t^3)\mathbf{j}$ m. Find the speed of the particle when $t = 2.5$ s.

6 Points A, B and C have position vectors $\mathbf{a} = -7\mathbf{i} + 11\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = 13\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

a Prove that the triangle ABC is isosceles.

b Find the position vector of point D such that the four points form a rhombus.

7 A particle moves with constant acceleration between the points A and B. At A, it has velocity $(4\mathbf{i} + 2\mathbf{j})$ m s^{-1}. At B, it has velocity $(7\mathbf{i} + 6\mathbf{j})$ m s^{-1}. It takes 10 seconds to move from A to B.

a Find the acceleration of the particle.

b Find the distance between A and B.

c Find the average velocity as the particle moves from A to B.

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8 A particle moves with velocity vector $\mathbf{v} = (1 - t)\mathbf{i} + (3t - 2)\mathbf{j}$.

Find the time at which the particle is moving parallel to the vector $\mathbf{i} + \mathbf{j}$.

Choose from these options.

A $t = 1$ **B** $t = \frac{2}{3}$ **C** $t = \frac{4}{3}$ **D** $t = \frac{3}{4}$

9 A particle of mass 0.3 kg starts from rest and moves under the action of a constant force $(6\mathbf{i}-2\mathbf{j})$ N. Find how long it takes to reach the speed of 12 m s^{-1}.

10 A helicopter is initially hovering above the helipad. It sets off with constant acceleration $(0.3\mathbf{i}+1.2\mathbf{j})$ m s^{-2}, where the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north, respectively. The helicopter is modelled as particle moving in two dimensions.

a Find the bearing on which the helicopter is travelling.

b Find the time at which the helicopter is 300 m from its initial position.

c Explain in everyday language the meaning of the modelling assumption that the helicopter moves in two dimensions.

11 Points P and Q have position vectors $\mathbf{p}=4\mathbf{i}-\mathbf{j}+11\mathbf{k}$ and $\mathbf{q}=3\mathbf{j}-\mathbf{k}$. S is the point on the line segment PQ such that $PS:SQ=3:2$. Fine the exact distance of S from the origin.

12 A particle of mass 2 kg moves in the plane under the action of the force $\mathbf{F}=(20\sin(2t)\,\mathbf{i}+30\cos(t)\mathbf{j})$ N. The particle is initially at rest at the origin.

Find the direction of motion of the particle after 5 seconds.

13 In this question, vectors $\mathbf{i}$ and $\mathbf{j}$ point due east and north, respectively.

A port is located at the origin. One ship starts from the port and moves with velocity $\mathbf{v}_1=(3\mathbf{i}+4\mathbf{j})$ km h^{-1}.

a Write down the position vector at time t hours.

At the same time, a second ship starts 18 km north of the port and moves with velocity $\mathbf{v}_2=(3\mathbf{i}-5\mathbf{j})$ km h^{-1}.

b Write down the position vector of the second ship at time t hours.

c Show that after half an hour, the distance between the two ships is 13.5 km.

d Show that the ships meet, and find the time when this happens.

e How long after the meeting are the ships 18 km apart?

14 A particle is initially at the point A, which has position vector $13.6\mathbf{i}$ m, with respect to an origin O. At the point A, the particle has velocity $(6\mathbf{i}+2.4\mathbf{j})$ m s^{-1}, and in its subsequent motion, it has a constant acceleration of $(-0.8\mathbf{i}+0.1\mathbf{j})$ m s^{-2}. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

a Find an expression for the velocity of the particle t seconds after it leaves A.

b Find an expression for the position vector of the particle, with respect to the origin O, t seconds after it leaves A

c Find the distance of the particle from the origin O when it is travelling in a north-westerly direction.

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15 A particle has velocity vector $\mathbf{v} = \begin{pmatrix} \sin t \\ e^{\frac{t}{2}} \end{pmatrix}$ and is initially at the origin.

Find the particle's position vector at time t.

Choose from these options.

A $\begin{pmatrix} 1-\cos t \\ 2e^{\frac{t}{2}}-2 \end{pmatrix}$ **B** $\begin{pmatrix} -\cos t \\ 2e^{\frac{t}{2}} \end{pmatrix}$ **C** $\begin{pmatrix} \cos t \\ \frac{1}{2}e^{\frac{t}{2}} \end{pmatrix}$ **D** $\begin{pmatrix} \cos t - 1 \\ \frac{1}{2}e^{\frac{t}{2}}-\frac{1}{2} \end{pmatrix}$

16 At time $t = 0$ two aircraft have position vectors $5\mathbf{j}$ and $7\mathbf{k}$. The first moves with constant velocity $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and the second with constant velocity $5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

a Write down the position vector of the first aircraft at time t.

Let d be the distance between the two aircraft at time t.

b Find an expression for d^2 in terms of t. Hence show that the two aircraft will not collide.

c Find the minimum distance between the two aircraft.

17 A position vector of a particle at time t seconds is given by $\mathbf{r} = (5\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}\text{ m})$.

a Find the Cartesian equation of the particle's trajectory.

b Find the maximum speed of the particle, and its position vector at the times when it has this maximum speed.

18 A particle of mass 3 kg moves on a horizontal surface under the action of the net force $\mathbf{F} = (36e^{-t}\mathbf{i} - 96e^{-2t}\mathbf{j})$ N. The particle is initially at the origin and has velocity $(-6\mathbf{i} + 20\mathbf{j})$ m s^{-1}. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north, respectively.

Find the distance of the particle from the origin at the time when it is travelling in the northerly direction.

Elevate

See Extension sheet 16 for questions on modelling rotation with vectors.